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SOME GROWTH ASPECTS OF COMPOSITE *P*-ADIC ENTIRE FUNCTIONS IN THE LIGHT OF THEIR (p, q)-TH RELATIVE ORDER AND (p, q)-TH RELATIVE TYPE

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Abstract. Let us consider that $\mathbb{K}$ be a complete ultrametric algebraically closed field and $\mathcal{A}(\mathbb{K})$ be the $\mathbb{K}$-algebra of entire functions on $\mathbb{K}$. In this paper we introduce the notions of (p, q)-th relative order and (p, q)-th relative type of p adic entire functions where p and q are any two positive integers and then study some growth properties of composite p adic entire functions in the light of their (p, q)-th relative order and (p, q)-th relative type. After that we show that (p, q) th relative order and (p, q)-th relative type are remain unchanged for derivatives under some certain conditions.

1. Introduction and Definitions

Let us consider $\mathbb{K}$ be an algebraically closed field of characteristic 0, complete with respect to a p-adic absolute value $|\cdot|$ (example $\mathbb{C}_p$). For any $\alpha \in \mathbb{K}$ and $R \in]0, +\infty[$, the closed disk $\{x \in \mathbb{K} : |x-\alpha| \leq R\}$ and the open disk $\{x \in \mathbb{K} : |x - \alpha| < R\}$ are denoted by $d(\alpha, R)$ and $d(\alpha, R^-)$ respectively. Also $C(\alpha, r)$ denotes the circle $\{x \in \mathbb{K} : |x - \alpha| = r\}$. Moreover $\mathcal{A}(\mathbb{K})$ represent the $\mathbb{K}$-algebra of analytic functions in $\mathbb{K}$ i.e. the set of power series with an infinite radius of convergence. For the most comprehensive study of analytic functions inside a disk or in the whole field $\mathbb{K}$, we refer the reader to the books [18, 23]. During the last several years the ideas of p-adic analysis have been studied from different aspects and many important results were gained (see [2] to [8], [11, 14, 15, 16, 17, 22]). We do not explain the standard definitions

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and notations in the theory of p-adic analysis as those are available in [12, 13]. Let $f \in \mathcal{A}(\mathbb{K})$ and $r > 0$, then we denote by $|f|(r)$ the number $\sup\{|f(x)|||x| = r\}$ where $|\cdot|(r)$ is a multiplicative norm on $\mathcal{A}(\mathbb{K})$. For $x \in [0, \infty)$ and $k \in \mathbb{N}$, we define $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$ and $\exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right)$ where $\mathbb{N}$ be the set of all positive integers. We also denote $\log^{[0]} x = x$ and $\exp^{[0]} x = x$. Throughout the paper, log denotes the Neperian logarithm. Taking this into account the order (resp. lower order) of an entire function $f \in \mathcal{A}(\mathbb{K})$ is given by (see [5])

$$\begin{matrix} \rho(f) \\ \lambda(f) \end{matrix} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[2]} |f|(r)}{\log r}.$$

The above definition of order (resp. lower order) does not seem to be feasible if an entire function $f \in \mathcal{A}(\mathbb{K})$ is of order zero. To over come this situation and in order to study the growth of f precisely, one may introduce the concept of logarithmic order (resp. logarithmic lower order) by increasing $\log^{+}$ once in the denominator following the classical definition of logarithmic order (see for example [10]). Therefore the logarithmic order $\rho_{\log}(f)$ and logarithmic lower order $\lambda_{\log}(f)$ of an entire function $f \in \mathcal{A}(\mathbb{K})$ are define as

$$\begin{matrix} \rho_{\log}(f) \\ \lambda_{\log}(f) \end{matrix} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[2]} |f|(r)}{\log^{[2]} r}.$$

Further one may introduce the following alternative definition of zero order (resp. zero lower order) of an entire function $f \in \mathcal{A}(\mathbb{K})$ when $\rho(f) = 0$(resp. $\lambda(f) = 0$)) in the following way:

Definition 1.1. Let $f \in \mathcal{A}(\mathbb{K})$ be such that $\rho(f) = 0$. Then the quantities$\underline{\rho}(f)$ and $\underline{\lambda}(f)$ of f are defined by:

$$\begin{matrix} \underline{\rho}(f) \\ \underline{\lambda}(f) \end{matrix} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log |f|(r)}{\log r},$$

and in this case one can easily verify that $\rho_{\log}(f) = 1$ and $\lambda_{\log}(f) = 1$.

Boussaf *et al.* [5] also introduce the definition of type (resp. lower type) of an entire function $f \in \mathcal{A}(\mathbb{K})$ which is also another type of growth indicator used for comparing the relative growth of two entire functions defined in $\mathcal{A}(\mathbb{K})$ having same non zero finite order in the following way:

$$\begin{matrix} \sigma(f) \\ \overline{\sigma}(f) \end{matrix} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log |f|(r)}{r^{\rho(f)}} \text{ where } 0 < \rho(f) < \infty.$$

Analogously for $0 < \lambda(f) < \infty$, one may give the definition of weak type $\tau(f)$ and the growth indicator $\overline{\tau}(f)$ of an entire function $f \in \mathcal{A}(\mathbb{K})$ in the following way:

$$\begin{matrix} \overline{\tau}(f) \\ \tau(f) \end{matrix} = \lim_{r \to +\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log |f|(r)}{r^{\lambda(f)}}.$$

Moreover the concept of (p, q)-th order (p and q are any two positive integers with $p \geq q$) is not new and was first introduced by Juneja *et al.* [19, 20]. In the line of Juneja *et al.* [19, 20], now we shall introduce the definitions of (p, q)-th order and (p, q)-th lower order respectively of an entire function $f \in \mathcal{A}(\mathbb{K})$ where $p, q \in \mathbb{N}$. In order to keep accordance with the definition of logarithmic order we will give a minor modification to the original definition of (p, q)-order introduced by Juneja et al. [19, 20].

Definition 1.2. Let $f \in \mathcal{A}(\mathbb{K})$ and $p, q \in \mathbb{N}$. Then the (p, q)-th order and (p, q)-th lower order of f are respectively define as:

$$\begin{matrix} \rho^{(p,q)}(f) \\ \lambda^{(p,q)}(f) \end{matrix} = \lim_{r \to +\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p]} |f|(r)}{\log^{[q]} r}.$$

These definitions extended the generalized order $\rho^{[l]}(f)$ and generalized lower order $\lambda^{[l]}(f)$ of $f \in \mathcal{A}(\mathbb{K})$ for each integer $l \geq 2$ since these correspond to the particular case $\rho^{[l]}(f) = \rho^{(l,1)}(f)$ and $\lambda^{[l]}(f) = \lambda^{(l,1)}(f)$. Clearly $\rho^{(2,1)}(f) = \rho(f)$ and $\lambda^{(2,1)}(f) = \lambda(f)$. The above definition avoids the restriction $p > q$ and gives the idea of generalized logarithmic order.

In this connection we just introduce the following definition which is analogous to a definition of Juneja *et al.* [19, 20]:

Definition 1.3. An entire function $f \in \mathcal{A}(\mathbb{K})$ is said to have index-pair (p, q) where p and $q \in \mathbb{N}$ if $b < \rho^{(p,q)}(f) < \infty$ and $\rho^{(p-1,q-1)}(f)$ is not a nonzero finite number, where $b = 1$ if $p = q$ and $b = 0$ for otherwise. Moreover if $0 < \rho^{(p,q)}(f) < \infty$, then

$$\begin{cases} \rho^{(p-n,q)}(f) = \infty & \text{for} \quad n < p, \\ \rho^{(p,q-n)}(f) = 0 & \text{for} \quad n < q, \\ \rho^{(p+n,q+n)}(f) = 1 & \text{for} \quad n = 1, 2, \cdots. \end{cases}$$

Similarly for $0 < \lambda^{(p,q)}(f) < \infty$, one can easily verify that

$$\begin{cases} \lambda^{(p-n,q)}(f) = \infty & \text{for} \quad n < p, \\ \lambda^{(p,q-n)}(f) = 0 & \text{for} \quad n < q, \\ \lambda^{(p+n,q+n)}(f) = 1 & \text{for} \quad n = 1, 2, \cdots. \end{cases}$$

An entire function $f \in \mathcal{A}(\mathbb{K})$ of index-pair (p,q) is said to be of regular (p,q) th growth if its (p,q) th order coincides with its (p,q) th lower order, otherwise f is said to be of irregular (p,q) th growth.

Next, to compare the growth of entire functions on $\mathbb{K}$ having the same (p,q)-th order, we give the definitions of (p,q)-th type and (p,q)-th lower type in the following manner:

Definition 1.4. The (p,q)-th type and the (p,q)-th lower type of entire function $f \in \mathcal{A}(\mathbb{K})$ having finite positive (p,q)-th order $\rho^{(p,q)}(f)$ $(b < \rho^{(p,q)}(f) < \infty)$ are defined as:

$$\begin{matrix} \sigma^{(p,q)}(f) \\ \overline{\sigma}^{(p,q)}(f) \end{matrix} = \lim_{r \to +\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-1]} |f|(r)}{\left(\log^{[q-1]} r\right)^{\rho^{(p,q)}(f)}},$$

where p,q are any two positive integers, $b = 1$ if $p = q$ and $b = 0$ for $p > q$. It is obvious that $0 \leq \overline{\sigma}^{(p,q)}(f) \leq \sigma^{(p,q)}(f) \leq \infty$.

Likewise, to compare the growth of entire functions on $\mathbb{K}$ having the same (p,q)-th lower order, one can also introduced the concepts of (p,q)-th weak type in the following manner:

Definition 1.5. The (p,q) th weak type of entire function $f \in \mathcal{A}(\mathbb{K})$ having finite positive (p,q) th tower order $\lambda^{(p,q)}(f)$ $(b < \lambda^{(p,q)}(f) < \infty)$ is defined as :

$$\tau^{(p,q)}(f) = \varliminf_{r \to +\infty} \frac{\log^{[p-1]} |f|(r)}{\left(\log^{[q-1]} r\right)^{\lambda^{(p,q)}(f)}}$$

where p,q are any two positive integers, $b = 1$ if $p = q$ and $b = 0$ for $p > q$.

Similarly one may define the growth indicator $\overline{\tau}^{(p,q)}(f)$ of an entire function $f \in \mathcal{A}(\mathbb{K})$ in the following way :

$$\overline{\tau}^{(p,q)}(f) = \varlimsup_{r \to +\infty} \frac{\log^{[p-1]} |f|(r)}{\left(\log^{[q-1]} r\right)^{\lambda^{(p,q)}(f)}} \ , \ b < \lambda^{(p,q)}(f) < \infty$$

where p,q are any two positive integers, $b = 1$ if $p = q$ and $b = 0$ for $p > q$. It is obvious that $0 \leq \tau^{(p,q)}(f) \leq \overline{\tau}^{(p,q)}(f) \leq \infty$.

However the concept of relative order in the complex context and as well as their technical advantages of not comparing with the growths of $\exp z$ was first introduced by Bernal [1]. In order to make some progress in the study of p-adic analysis, recently Biswas [9] introduce the definition of relative order and relative lower order of entire function $f \in \mathcal{A}(\mathbb{K})$ with respect to another entire function $g \in \mathcal{A}(\mathbb{K})$ in the following way:

$$\begin{matrix} \rho_g(f) \\ \lambda_g(f) \end{matrix} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log \widehat{|g|}(|f|(r))}{\log r}.$$

Further the function $f \in \mathcal{A}(\mathbb{K})$, for which relative order and relative lower order with respect to another function $g \in \mathcal{A}(\mathbb{K})$ are the same is called a function of regular relative growth with respect to g. Otherwise, f is said to be irregular relative growth.with respect to g.

In the case of relative order, it therefore seems reasonable to define suitably the (p,q)-th relative order as well as (p,q)-th relative type of entire function belonging to $\mathcal{A}(\mathbb{K})$ and to investigate some of its basic properties, which we attempt in this paper. With this in view we introduce the following definitions of (p,q)-th relative order $\rho_g^{(p,q)}(f)$ and (p,q)-th relative lower order $\lambda_g^{(p,q)}(f)$ of an entire function $f \in \mathcal{A}(\mathbb{K})$ with respect to another entire function $g \in \mathcal{A}(\mathbb{K})$, in the light of index-pair. Our next definition avoids the restriction $p > q$.

Definition 1.6. Let f, $g \in \mathcal{A}(\mathbb{K})$. Also let the index-pair of f and g are (m,q) and (m,p), respectively, where p, q, m are positive integers. Then the (p,q)-th relative order $\rho_g^{(p,q)}(f)$ and (p,q)-th relative lower order $\lambda_g^{(p,q)}(f)$ of f with respect to g are defined as

$$\begin{matrix} \rho_g^{(p,q)}(f) \\ \lambda_g^{(p,q)}(f) \end{matrix} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p]} \widehat{|g|}(|f|(r))}{\log^{[q]} r} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p]} \widehat{|g|}(r)}{\log^{[q]} \widehat{|f|}(r)}.$$

Further for any $f \in \mathcal{A}(\mathbb{K})$ for which (p,q)-th relative order and (p,q)-th relative lower order with respect to $g \in \mathcal{A}(\mathbb{K})$ are the same is called a function of regular relative (p,q) growth with respect to g. Otherwise, f is said to be irregular relative (p,q) growth with respect to g.

In the case of relative order, one may also introduce the definition of relative index-pair in the following manner:

Definition 1.7. An entire function $f \in \mathcal{A}(\mathbb{K})$ is said to have relative index-pair (p,q) with respect to another entire function $g \in \mathcal{A}(\mathbb{K})$ where p and $q \in \mathbb{N}$ if $b < \rho_g^{(p,q)}(f) < \infty$ and $\rho_g^{(p-1,q-1)}(f)$ is not a nonzero finite number, where $b = 1$ if $p = q$ and $b = 0$ for otherwise. Moreover if

$0<\rho_g^{(p,q)}(f)<\infty$, then

$$\begin{cases} \rho_g^{(p-n,q)}(f)=\infty & \text{for} \quad n<p, \\ \rho_g^{(p,q-n)}(f)=0 & \text{for} \quad n<q, \\ \rho_g^{(p+n,q+n)}(f)=1 & \text{for} \quad n=1,2,\cdots . \end{cases}$$

Similarly for $0<\lambda_g^{(p,q)}(f)<\infty$, one can easily verify that

$$\begin{cases} \lambda_g^{(p-n,q)}(f)=\infty & \text{for} \quad n<p, \\ \lambda_g^{(p,q-n)}(f)=0 & \text{for} \quad n<q, \\ \lambda_g^{(p+n,q+n)}(f)=1 & \text{for} \quad n=1,2,\cdots . \end{cases}$$

Now in order to refine the above growth scale, we introduce the definitions of another growth indicators, called (p,q) -th relative type and (p,q) -th relative lower type respectively of p adic entire function with respect to another p adic entire function in the light of their index-pair which is as follows:

Definition 1.8. Let f, $g\in\mathcal{A}(\mathbb{K})$. Also let the index-pair of f and g are (m,q) and (m,p), respectively, where p,q,m are positive integers. The (p,q) -th relative type and (p,q) -th relative lower type of f with respect to g having finite positive (p,q)-th relative order $\rho_g^{(p,q)}(f)$ $\left(a<\rho_g^{(p,q)}(f)<\infty\right)$ are defined as :

$$\begin{matrix}\sigma_g^{(p,q)}(f) \\ \overline{\sigma}_g^{(p,q)}(f)\end{matrix} = \lim_{r\to+\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log^{[p-1]}\widehat{|g|}(|f|(r))}{\left(\log^{[q-1]} r\right)^{\rho_g^{(p,q)}(f)}},$$

where p,q are any two positive integers, $b=1$ if $p=q$ and $b=0$ for $p>q$. It is obvious that $0\leq\overline{\sigma}_g^{(p,q)}(f)\leq\sigma_g^{(p,q)}(f)\leq\infty$.

Analogously, to determine the relative growth of two entire functions on $\mathbb{K}$ and having same non zero finite (p,q)-th relative lower order with respect to another entire function belonging to $\mathcal{A}(\mathbb{K})$, one can introduced the definition of (p,q) -th relative weak type of an entire function $f\in\mathcal{A}(\mathbb{K})$ with respect to $g\in\mathcal{A}(\mathbb{K})$ of finite positive (p,q) -th relative lower order $\lambda_g^{(p,q)}(f)$ in the following way:

Definition 1.9. Let f, $g\in\mathcal{A}(\mathbb{K})$. Also let the index-pair of f and g are (m,q) and (m,p), respectively, where p,q,m are positive integers. The (p,q) -th relative weak type and the growth indicator $\overline{\tau}_g^{(p,q)}(f)$ of

f with respect to g having finite positive (p,q)-th relative lower order $\lambda_g^{(p,q)}(f)$ $\left(a < \lambda_g^{(p,q)}(f) < \infty\right)$ are defined as :

$$\begin{matrix}\overline{\tau}_g^{(p,q)}(f) \\ \tau_g^{(p,q)}(f)\end{matrix} = \lim_{r\to+\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log^{[p-1]} \widehat{|g|}\left(|f|(r)\right)}{\left(\log^{[q-1]} r\right)^{\lambda_g^{(p,q)}(f)}},$$

where p, q are any two positive integers, $b = 1$ if $p = q$ and $b = 0$ for $p > q$. It is obvious that $0 \le \tau_g^{(p,q)}(f) \le \overline{\tau}_g^{(p,q)}(f) \le \infty$.

For any two entire functions $f \in \mathcal{A}(\mathbb{K})$ and $g \in \mathcal{A}(\mathbb{K})$ the ratio $\frac{|f|(r)}{|g|(r)}$ as $r \to +\infty$ is called the growth of f with respect to g in terms of their multiplicative norm. Actually the studies of the growths of composite p adic entire functions in the light of their relative growth indicators are the prime concern of this paper. After discussing some growth properties of composite p adic entire functions in the light of their (p,q)-th relative order and (p,q)-th relative type, we show that (p,q) th relative order and (p,q)-th relative type are remain unchanged for derivatives under some certain conditions.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

The following lemma due to A. Escassut [12] which can also be found in [5] or [8].

Lemma 2.1. *Let $f, g \in \mathcal{A}(\mathbb{K})$. Then for all sufficiently large values of r*

$$|f \circ g|(r) = |f|\left(|g|(r)\right).$$

Lemma 2.2. *Let $f(x) = \sum\limits_{n=0}^{+\infty} a_n x^n \in \mathcal{A}(\mathbb{K})$ and let a, b be positive with $a < b$ and $b > 1$. Then when r is large enough, one has*

$$r^a |f|(r) \le |f|\left(r^b\right).$$

Proof. Since $f(x) = \sum\limits_{n=0}^{+\infty} a_n x^n \in \mathcal{A}(\mathbb{K})$. Then for all $r > 0$ we have $|f|(r) = \sup\limits_{n\ge0} \{|a_n| r^n\}$.

Now

$$|f|\left(r^{b}\right)=\sup_{n\geq 0}\left\{|a_{n}|\left(r\right)^{bn}\right\}. \tag{2.1}$$

Also

$$\begin{aligned} r^{a}|f|\left(r\right) &= r^{a}\sup_{n\geq 0}\left\{|a_{n}|r^{n}\right\}, \\ i.e.,\ r^{a}|f|\left(r\right) &= \sup_{n\geq 0}\left\{|a_{n}|\left(r\right)^{a+n}\right\}. \end{aligned} \tag{2.2}$$

As we take the supremum value for large r, therefore $n\neq 0$ and as $a<b$ so $a+n\leq bn$ holds. Hence from (2.1) and (2.2) we get that $r^{a}|f|\left(r\right)\leq|f|\left(r^{b}\right)$.

Thus the lemma follows. □

3. Main Results

In this section we present the main results of the paper.

THEOREM 3.1. *Let f, g, $h\in\mathcal{A}\left(\mathbb{K}\right)$. Also let the relative index pair of f with respect to h and the index pair of g are (p,q) and (m,n) respectively where $p,q,m,n\in\mathbb{N}$. Then*

(i) *the relative index-pair of $f\circ g$ is (p,n) when $q=m$ and either $\lambda_{h}^{(p,q)}\left(f\right)>0$ or $\lambda^{(m,n)}\left(g\right)>0$. Also*

(a) $\lambda_{h}^{(p,q)}(f)\rho^{(m,n)}\left(g\right)\leq\rho_{h}^{(p,n)}\left(f\circ g\right)\rho_{h}^{(p,q)}\left(f\right)\rho^{(m,n)}\left(g\right)$ *if* $\lambda_{h}^{(p,q)}\left(f\right)>0$ *and*

(b) $\rho_{h}^{(p,q)}\left(f\right)\lambda^{(m,n)}\left(g\right)\leq\rho_{h}^{(p,n)}\left(f\circ g\right)\rho_{h}^{(p,q)}\left(f\right)\rho^{(m,n)}\left(g\right)$ *if* $\lambda^{(m,n)}\left(g\right)>0$;

(ii) *the relative index-pair of $f\circ g$ is $(p,q+n-m)$ when $q>m$ and either $\lambda_{h}^{(p,q)}\left(f\right)>0$ or $\lambda^{(m,n)}\left(g\right)>0$. Also*

(a) $\lambda_{h}^{(p,q)}\left(f\right)\leq\rho_{h}^{(p,q+n-m)}\left(f\circ g\right)\rho_{h}^{(p,q)}\left(f\right)$ *if* $\lambda_{h}^{(p,q)}\left(f\right)>0$ *and*

(b) $\rho_{h}^{(p,q+n-m)}\left(f\circ g\right)=\rho_{h}^{(p,q)}\left(f\right)$ *if* $\lambda^{(m,n)}\left(g\right)>0$;

(iii) *the relative index-pair of $f\circ g$ is $(p+m-q,n)$ when $q<m$ and either $\lambda_{h}^{(p,q)}(f)>0$ or $\lambda^{(m,n)}(g)>0$. Also*

(a) $\rho_{h}^{(p+m-q,n)}\left(f\circ g\right)=\rho^{(m,n)}\left(g\right)$ *if* $\lambda_{h}^{(p,q)}\left(f\right)>0$ *and*

(b) $\lambda^{(m,n)}\left(g\right)\leq\rho_{h}^{(p+m-q,n)}\left(f\circ g\right)\rho^{(m,n)}\left(g\right)$ *if* $\lambda^{(m,n)}\left(g\right)>0$.

Proof. In view of Lemma 2.1, it follows for all sufficiently large positive numbers of r that

$$\log^{[p]}\widehat{|h|}\left(|f\circ g|\left(r\right)\right)\geq\left(\lambda_{h}^{(p,q)}\left(f\right)-\varepsilon\right)\log^{[q]}|g|\left(r\right) \tag{3.1}$$

and also for a sequence of positive numbers of r tending to infinity we get that

$$\log^{[p]} \widehat{|h|} \left(|f \circ g| (r)\right) \geq \left(\rho_h^{(p,q)} (f) - \varepsilon\right) \log^{[q]} |g| (r) . \tag{3.2}$$

Similarly, we have for all sufficiently large positive numbers of r that

$$\log^{[p]} \widehat{|h|} \left(|f \circ g| (r)\right) \left(\rho_h^{(p,q)} (f) + \varepsilon\right) \log^{[q]} |g| (r) . \tag{3.3}$$

Now the following three cases may arise:

Case I. Let $q = m$. In this case we have from (3.3) for all sufficiently large positive numbers of r that

$$\log^{[p]} \widehat{|h|} \left(|f \circ g| (r)\right) \left(\rho_h^{(p,q)} (f) + \varepsilon\right) \left(\rho^{(m,n)} (g) + \varepsilon\right) \log^{[n]} r,$$

$$i.e., \lim_{r \to +\infty} \frac{\log^{[p]} \widehat{|h|} \left(|f \circ g| (r)\right)}{\log^{[n]} r} \rho_h^{(p,q)} (f) \, \rho^{(m,n)} (g) . \tag{3.4}$$

Also from (3.1), we obtain for a sequence of positive numbers of r tending to infinity that

$$\log^{[p]} \widehat{|h|} \left(|f \circ g| (r)\right) \geq \left(\lambda_h^{(p,q)} (f) - \varepsilon\right) \left(\rho^{(m,n)} (g) - \varepsilon\right) \log^{[n]} r,$$

$$i.e., \overline{\lim_{r \to +\infty}} \frac{\log^{[p]} \widehat{|h|} \left(|f \circ g| (r)\right)}{\log^{[n]} r} \geq \lambda_h^{(p,q)} (f) \, \rho^{(m,n)} (g) . \tag{3.5}$$

Moreover, we have from (3.2) for a sequence of positive numbers of r tending to infinity that

$$\log^{[p]} \widehat{|h|} \left(|f \circ g| (r)\right) \geq \left(\rho_h^{(p,q)} (f) - \varepsilon\right) \left(\lambda^{(m,n)} (g) - \varepsilon\right) \log^{[n]} r,$$

$$i.e., \overline{\lim_{r \to +\infty}} \frac{\log^{[p]} \widehat{|h|} \left(|f \circ g| (r)\right)}{\log^{[n]} r} \geq \rho_h^{(p,q)} (f) \, \lambda^{(m,n)} (g) . \tag{3.6}$$

Therefore from (3.4) and (3.5), we get for $\lambda_h^{(p,q)} (f) > 0$ that

$$\lambda_h^{(p,q)} (f) \, \rho^{(m,n)} (g) \leq \overline{\lim_{r \to +\infty}} \frac{\log^{[p]} \widehat{|h|} \left(|f \circ g| (r)\right)}{\log^{[n]} r} \rho_h^{(p,q)} (f) \, \rho^{(m,n)} (g) ,$$

$$i.e., \lambda_h^{(p,q)} (f) \, \rho^{(m,n)} (g) \leq \rho_h^{(p,n)} (f \circ g) \, \rho_h^{(p,q)} (f) \, \rho^{(m,n)} (g) . \tag{3.7}$$

Likewise, from (3.4) and (3.6) we obtain for $\lambda^{(m,n)} (g) > 0$ that

$$\rho_h^{(p,q)} (f) \, \lambda^{(m,n)} (g) \leq \overline{\lim_{r \to +\infty}} \frac{\log^{[p]} \widehat{|h|} \left(|f \circ g| (r)\right)}{\log^{[n]} r} \rho_h^{(p,q)} (f) \, \rho^{(m,n)} (g) ,$$

$$(3.8) \qquad i.e., \rho_h^{(p,q)}(f)\,\lambda^{(m,n)}(g) \leq \rho_h^{(p,n)}(f \circ g)\,\rho_h^{(p,q)}(f)\,\rho^{(m,n)}(g)\,.$$

Also from (3.7) and (3.8) one can easily verify that $\rho_h^{(p-1,n)}(f \circ g) = \infty$, $\rho_h^{(p,n-1)}(f \circ g) = 0$ and $\rho_h^{(p+1,n+1)}(f \circ g) = 1$ and therefore we obtain that the relative index-pair of $f \circ g$ is (p,n) when $q = m$ and either $\lambda_h^{(p,q)}(f) > 0$ or $\lambda^{(m,n)}(g) > 0$ and thus the first part of the theorem is established.

Case II. Let $q > m$. Now we obtain from (3.3) for all sufficiently large positive numbers of r that

$$\log^{[p]} \widehat{|h|}\,(|f \circ g|\,(r) \left(\rho_h^{(p,q)}(f) + \varepsilon\right) \log^{[q-m]} \log^{[m],} |g|\,(r)\,,$$

$$i.e., \; \log^{[p]} \widehat{|h|}\,(|f \circ g|\,(r) \left(\rho_h^{(p,q)}(f) + \varepsilon\right) \log^{[q-m]} \left[\left(\rho^{(m,n)}(g) + \varepsilon\right) \log^{[n]} r\right],$$

$$i.e., \log^{[p]} \widehat{|h|}\,(|f \circ g|\,(r) \left(\rho_h^{(p,q)}(f) + \varepsilon\right) \log^{[q+n-m]} r + O(1),$$

$$(3.9) \qquad i.e., \lim_{r \to +\infty} \frac{\log^{[p]} \widehat{|h|}\,(|f \circ g|\,(r))}{\log^{[q+n-m]} r} \rho_h^{(p,q)}(f)\,.$$

Also from (3.1), we have for a sequence of positive numbers of r tending to infinity that

$$\log^{[p]} \widehat{|h|}\,(|f \circ g|\,(r)) \geq \left(\lambda_h^{(p,q)}(f) - \varepsilon\right) \log^{[q-m]} \left[\left(\rho^{(m,n)}(g) - \varepsilon\right) \log^{[n]}(r)\right],$$

$$i.e., \log^{[p]} \widehat{|h|}\,(|f \circ g|\,(r)) \geq \left(\lambda_h^{(p,q)}(f) - \varepsilon\right) \log^{[q-m+n]} r + O(1),$$

$$(3.10) \qquad i.e., \overline{\lim_{r \to +\infty}} \frac{\log^{[p]} \widehat{|h|}\,(|f \circ g|\,(r))}{\log^{[q+n-m]} r} \geq \lambda_h^{(p,q)}(f)\,.$$

Further, we get from (3.2) for a sequence of positive numbers of r tending to infinity that

$$\log^{[p]} \widehat{|h|}\,(|f \circ g|\,(r)) \geq \left(\rho_h^{(p,q)}(f) - \varepsilon\right) \log^{[q-m]} \left[\left(\lambda^{(m,n)}(g) - \varepsilon\right) \log^{[n]}(r)\right],$$

$$i.e., \log^{[p]} \widehat{|h|}\,(|f \circ g|\,(r)) \geq \left(\rho_h^{(p,q)}(f) - \varepsilon\right) \log^{[q+n-m]} r + O(1),$$

$$(3.11) \qquad i.e., \overline{\lim_{r \to +\infty}} \frac{\log^{[p]} \widehat{|h|}\,(|f \circ g|\,(r))}{\log^{[q+n-m]} r} \geq \rho_h^{(p,q)}(f)\,.$$

Therefore from (3.9) and (3.10), we get for $\lambda_h^{(p,q)}(f) > 0$ that

$$\lambda_h^{(p,q)}(f) \leq \overline{\lim_{r\to+\infty}} \frac{\log^{[p]} \widehat{|h|}(|f\circ g|(r))}{\log^{[q+n-m]} r} \rho_h^{(p,q)}(f),$$

$$\text{(3.12)} \quad i.e., \ \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q+n-m)}(f\circ g)\,\rho_h^{(p,q)}(f).$$

Likewise, from (3.9) and (3.11) we get for $\lambda^{(m,n)}(g) > 0$ that

$$\rho_h^{(p,q)}(f) \leq \overline{\lim_{r\to+\infty}} \frac{\log^{[p]} \widehat{|h|}(|f\circ g|(r))}{\log^{[q+n-m]} r} \leq \rho_h^{(p,q)}(f),$$

$$\text{(3.13)} \quad i.e., \ \rho_h^{(p,q+n-m)}(f\circ g) = \rho_h^{(p,q)}(f).$$

Hence from (3.12) and (3.13) one can easily verify that $\rho_h^{(p-1,q+n-m)}(f\circ g) = \infty$, $\rho_h^{(p,q+n-m-1)}(f\circ g) = 0$ and $\rho_h^{(p+1,q+n-m+1)}(f\circ g) = 1$ and therefore we get that the relative index-pair of $f\circ g$ is $(p, q+n-m)$ when $q > m$ and either $\lambda_h^{(p,q)}(f) > 0$ or $\lambda^{(m,n)}(g) > 0$ and thus the second part of the theorem follows.

Case III. Let $q < m$. Then we obtain from (3.3) for all sufficiently large positive numbers of r that

$$\log^{[p+m-q]} \widehat{|h|}(|f\circ g|(r)) \quad \log^{[m]} |g|(r) + O(1),$$
$$i.e., \log^{[p+m-q]} \widehat{|h|}(|f\circ g|(r)) \quad \left(\rho^{(m,n)}(g) + \varepsilon\right) \log^{[n]} r + O(1),$$

$$\text{(3.14)} \quad i.e., \ \lim_{r\to+\infty} \frac{\log^{[p+m-q]} \widehat{|h|}(|f\circ g|(r))}{\log^{[n]} r} \rho^{(m,n)}(g).$$

Also from (3.1) we have for a sequence of positive numbers of r tending to infinity that

$$\log^{[p+m-q]} \widehat{|h|}(|f\circ g|(r)) \geq \log^{[m]} |g|(r) + O(1),$$
$$i.e., \ \log^{[p+m-q]} \widehat{|h|}(|f\circ g|(r)) \geq \left(\rho^{(m,n)}(g) - \varepsilon\right) \log^{[n]} r + O(1),$$

$$\text{(3.15)} \quad \overline{\lim_{r\to+\infty}} \frac{\log^{[p+m-q]} \widehat{|h|}(|f\circ g|(r))}{\log^{[n]} r} \geq \rho^{(m,n)}(g).$$

Further, we get from (3.2) for a sequence of positive numbers of r tending to infinity that

$$\log^{[p+m-q]} \widehat{|h|}(|f\circ g|(r)) \geq \log^{[m]} |g|(r) + O(1),$$
$$i.e., \ \log^{[p+m-q]} \widehat{|h|}(|f\circ g|(r)) \geq \left(\lambda^{(m,n)}(g) - \varepsilon\right) \log^{[n]} r + O(1),$$

$$\overline{\lim_{r\to+\infty}}\frac{\log^{[p+m-q]}\widehat{|h|}\left(|f\circ g|\left(r\right)\right)}{\log^{[n]}r}\geq\lambda^{(m,n)}\left(g\right). \tag{3.16}$$

Therefore from (3.14) and (3.15), we obtain for $\lambda_h^{(p,q)}(f)>0$ that

$$\rho^{(m,n)}\left(g\right)\leq\overline{\lim_{r\to+\infty}}\frac{\log^{[p+m-q]}\widehat{|h|}\left(|f\circ g|\left(r\right)\right)}{\log^{[n]}r}\rho^{(m,n)}\left(g\right),$$

$$i.e.,\ \rho_h^{(p+m-q,n)}\left(f\circ g\right)=\rho^{(m,n)}\left(g\right). \tag{3.17}$$

Similarly, from (3.14) and (3.16) we get for $\lambda^{(m,n)}(g)>0$ that

$$\lambda^{(m,n)}\left(g\right)\leq\overline{\lim_{r\to+\infty}}\frac{\log^{[p+m-q]}\widehat{|h|}\left(|f\circ g|\left(r\right)\right)}{\log^{[n]}r}\rho^{(m,n)}\left(g\right),$$

$$i.e.,\ \lambda^{(m,n)}\left(g\right)\leq\rho_h^{(p+m-q,n)}\left(f\circ g\right)\rho^{(m,n)}\left(g\right). \tag{3.18}$$

So from (3.17) and (3.18) one can easily verify that $\rho_h^{(p+m-q-1,n)}(f\circ g)=\infty$, $\rho_h^{(p+m-q,n-1)}(f\circ g)=0$ and $\rho_h^{(p+m-q+1,n+1)}(f\circ g)=1$ and therefore we obtain that the relative index-pair of $f\circ g$ is $(p+m-q,n)$ when $q<m$ and either $\lambda_h^{(p,q)}(f)>0$ or $\lambda^{(m,n)}(g)>0$ and thus the third part of the theorem is established. □

In the line of Theorem 3.1 one can easily deduce the conclusion of the following theorem and so its proof is omitted.

Theorem 3.2. *Let f, g, $h\in\mathcal{A}(\mathbb{K})$. Also let the relative index pair of f with respect to h and the index pair of g are (p,q) and (m,n) respectively where $p,q,m,n\in\mathbb{N}$. Then*

(i) $\lambda_h^{(p,q)}(f)\lambda^{(m,n)}(g)$
$\leq\lambda_h^{(p,n)}(f\circ g)\min\left\{\rho_h^{(p,q)}(f)\lambda^{(m,n)}(g),\lambda_h^{(p,q)}(f)\rho^{(m,n)}(g)\right\}$ *if* $q=m$, $\lambda_h^{(p,q)}(f)>0$ *and* $\lambda^{(m,n)}(g)>0$;

(ii) $\lambda_h^{(p,q+n-m)}(f\circ g)=\lambda_h^{(p,q)}(f)$ *if* $q>m$, $\lambda_h^{(p,q)}(f)>0$ *and* $\lambda^{(m,n)}(g)>0$;

(iii) $\lambda_h^{(p+m-q,n)}(f\circ g)=\lambda^{(m,n)}(g)$ *if* $q<m$, $\lambda_h^{(p,q)}(f)>0$ *and* $\lambda^{(m,n)}(g)>0$.

Corollary 3.3. *Let f, g, $h\in\mathcal{A}(\mathbb{K})$. Also let the relative index pair of f with respect to h and the index pair of g are $(p-l,m-l)$*

and (m,n) *respectively where* $p,q,m,n \in \mathbb{N}$ *such that* $p-l>0$ *and* $m-l>0$.*Then*

$$\rho_h^{(p,n)}(f\circ g)=\rho^{(m,n)}(g) \text{ and } \lambda_h^{(p,n)}(f\circ g)=\lambda^{(m,n)}(g).$$

Proof. In view of Definition 1.7 $\rho_h^{(p,m)}(f)=\lambda_h^{(p,m)}(f)=1$. Therefore the conclusion of above corollary immediately follows from the first part of Theorem 3.1 and Theorem 3.2. □

THEOREM 3.4. *Let* $f,\ g\in\mathcal{A}(\mathbb{K})$ *have index-pairs* (p,q) *and* (m,n) *respectively where* $p,q,m,n\in\mathbb{N}$. *Then*

(i) *the index-pair of* $f\circ g$ *is* (p,n) *when* $q=m$ *and either* $\lambda^{(p,q)}(f)>0$ *or* $\lambda^{(m,n)}(g)>0$. *Also*
- (a) $\lambda^{(p,q)}(f)\rho^{(m,n)}(g)\le\rho^{(p,n)}(f\circ g)\rho^{(p,q)}(f)\rho^{(m,n)}(g)$ *if* $\lambda^{(p,q)}(f)>0$ *and*
- (b) $\rho^{(p,q)}(f)\lambda^{(m,n)}(g)\le\rho^{(p,n)}(f\circ g)\rho^{(p,q)}(f)\rho^{(m,n)}(g)$ *if* $\lambda^{(m,n)}(g)>0$;

(ii) *the index-pair of* $f\circ g$ *is* $(p,q+n-m)$ *when* $q>m$ *and either* $\lambda^{(p,q)}(f)>0$ *or* $\lambda^{(m,n)}(g)>0$. *Also*
- (a) $\lambda^{(p,q)}(f)\le\rho^{(p,q+n-m)}(f\circ g)\rho^{(p,q)}(f)$ *if* $\lambda^{(p,q)}(f)>0$ *and*
- (b) $\rho^{(p,q+n-m)}(f\circ g)=\rho^{(p,q)}(f)$ *if* $\lambda^{(m,n)}(g)>0$;

(iii) *the index-pair of* $f\circ g$ *is* $(p+m-q,n)$ *when* $q<m$ *and either* $\lambda^{(p,q)}(f)>0$ *or* $\lambda^{(m,n)}(g)>0$. *Also*
- (a) $\rho^{(p+m-q,n)}(f\circ g)=\rho^{(m,n)}(g)$ *if* $\lambda^{(p,q)}(f)>0$ *and*
- (b) $\lambda^{(m,n)}(g)\le\rho^{(p+m-q,n)}(f\circ g)\rho^{(m,n)}(g)$ *if* $\lambda^{(m,n)}(g)>0$.

THEOREM 3.5. *Let* $f,\ g\in\mathcal{A}(\mathbb{K})$ *have index-pairs* (p,q) *and* (m,n) *respectively where* $p,q,m,n\in\mathbb{N}$. *Then*

(i) $\lambda^{(p,q)}(f)\lambda^{(m,n)}(g)$
$\le\lambda^{(p,n)}(f\circ g)\min\left\{\rho^{(p,q)}(f)\lambda^{(m,n)}(g),\lambda^{(p,q)}(f)\rho^{(m,n)}(g)\right\}$ *if* $q=m$, $\lambda^{(p,q)}(f)>0$ *and* $\lambda^{(m,n)}(g)>0$;

(ii) $\lambda^{(p,q+n-m)}(f\circ g)=\lambda^{(p,q)}(f)$ *if* $q>m$, $\lambda^{(p,q)}(f)>0$ *and* $\lambda^{(m,n)}(g)>0$;

(iii) $\lambda^{(p+m-q,n)}(f\circ g)=\lambda^{(m,n)}(g)$ *if* $q<m$, $\lambda^{(p,q)}(f)>0$ *and* $\lambda^{(m,n)}(g)>0$.

Reasoning similarly as in the proofs of the Theorem 3.1 and Theorem 3.2 one can easily deduce the conclusions of the above two theorems, and so their proofs are omitted.

REMARK 3.6. In the complex context, similar nature of results as proved in Theorem 3.4 were established by Hong-Yan Xu *et al.* [24].

Remark 3.7. Let $f,\ g\in\mathcal{A}(\mathbb{K})$ have index-pairs $(p-l,m-l)$ and (m,n) respectively where $p,q,m,n,l\in\mathbb{N}$ such that $p-l>0$ and $m-l>0$.Then in view of Definition 1.3, Theorem 3.4 and Theorem 3.5 $\rho^{(p,n)}(f\circ g)=\rho^{(m,n)}(g)$ and $\lambda^{(p,n)}(f\circ g)=\lambda^{(m,n)}(g)$.

Remark 3.8. In view of Definition 1.1, Theorem 3.4 and Theorem 3.5, it is evident that $\rho(f\circ g)$ and $\lambda(f\circ g)$ are both finite when $\rho(f)<\infty$ and $\rho(g)=0$. Further if $\rho(f)=0$ and $f,g\in\mathcal{A}(\mathbb{K})$ be transcendental, then in view of Corollary 3.3 $\rho(f\circ g)=\rho(g)$ which improve Theorem 3 of [5].

Corollary 3.9. *Let $f,\ g,\ h\in\mathcal{A}(\mathbb{K})$. Also let the relative index pair of f with respect to h and the index pair of g are $(p-1,m-1)$ and (m,n) respectively where $p,q,m,n\in\mathbb{N}$ such that $p>1$ and $m>1$.Then*

$$\lambda_h^{(p-1,q-1)}(f)\,\sigma^{(m,n)}(g)\le\sigma_h^{(p,n)}(f\circ g)\le\rho_h^{(p-1,q-1)}(f)\,\sigma^{(m,n)}(g)$$

and

$$\lambda^{(p-1,q-1)}(f)\,\overline{\sigma}^{(m,n)}(g)\le\overline{\sigma}^{(p,n)}(f\circ g)\le\rho^{(p-1,q-1)}(f)\,\overline{\sigma}^{(m,n)}(g)$$

Proof. In view of Lemma 2.1 and Corollary 3.3, we get that

$$\begin{aligned}
&{\sigma_h}^{(p,n)}(f\circ g)\\
&=\overline{\lim_{r\to+\infty}}\frac{\log^{[p-1]}\widehat{|h|}(|f\circ g|(r))}{\left(\log^{[n-1]}r\right)^{\rho_h^{(p,n)}(f\circ g)}}\\
&\le\overline{\lim_{r\to+\infty}}\frac{\log^{[p-1]}\widehat{|h|}(|f|(|g|(r)))}{\log^{[m-1]}|g|(r)}\cdot\overline{\lim_{r\to+\infty}}\frac{\log^{[m-1]}|g|(r)}{\left(\log^{[n-1]}r\right)^{\rho^{(m,n)}(g)}}
\end{aligned}$$

$$=\rho_h^{(p-1,q-1)}(f)\,\sigma^{(m,n)}(g).\tag{3.19}$$

Similarly

$$\begin{aligned}
&{\sigma_h}^{(p,n)}(f\circ g)\\
&=\overline{\lim_{r\to+\infty}}\frac{\log^{[p-1]}\widehat{|h|}(|f\circ g|(r))}{\left(\log^{[n-1]}r\right)^{\rho_h^{(p,n)}(f\circ g)}}\\
&\ge\underline{\lim_{r\to+\infty}}\frac{\log^{[p-1]}\widehat{|h|}(|f|(|g|(r)))}{\log^{[m-1]}|g|(r)}\cdot\overline{\lim_{r\to+\infty}}\frac{\log^{[m-1]}|g|(r)}{\left(\log^{[n-1]}r\right)^{\rho^{(m,n)}(g)}}
\end{aligned}$$

$$=\lambda_h^{(p-1,q-1)}(f)\,\sigma^{(m,n)}(g).\tag{3.20}$$

Hence the first part of corollary follows from (3.19) and (3.20).

Again

$$\overline{\sigma}_h^{(p,n)}(f\circ g) = \underline{\lim}_{r\to+\infty}\frac{\log^{[p-1]}\widehat{|h|}\left(|f\circ g|(r)\right)}{\left(\log^{[n-1]}r\right)^{\rho_h^{(p,n)}(f\circ g)}}$$

$$\geq \underline{\lim}_{r\to+\infty}\frac{\log^{[p-1]}\widehat{|h|}\left(|f|\left(|g|(r)\right)\right)}{\log^{[m-1]}|g|(r)}\cdot\underline{\lim}_{r\to+\infty}\frac{\log^{[m-1]}|g|(r)}{\left(\log^{[n-1]}r\right)^{\rho^{(m,n)}(g)}}$$

$$= \lambda_h^{(p-1,q-1)}(f)\,\overline{\sigma}^{(m,n)}(g)\,. \tag{3.21}$$

Also

$$\overline{\sigma}_h^{(p,n)}(f\circ g) = \underline{\lim}_{r\to+\infty}\frac{\log^{[p-1]}\widehat{|h|}\left(|f\circ g|(r)\right)}{\left(\log^{[n-1]}r\right)^{\rho_h^{(p,n)}(f\circ g)}}$$

$$\leq \overline{\lim}_{r\to+\infty}\frac{\log^{[p-1]}\widehat{|h|}\left(|f|\left(|g|(r)\right)\right)}{\log^{[m-1]}|g|(r)}\cdot\underline{\lim}_{r\to+\infty}\frac{\log^{[m-1]}|g|(r)}{\left(\log^{[n-1]}r\right)^{\rho^{(m,n)}(g)}}$$

$$= \rho_h^{(p-1,q-1)}(f)\,\overline{\sigma}^{(m,n)}(g)\,. \tag{3.22}$$

Therefore the second part of corollary follows from (3.21) and (3.22). These complete the proofs. □

Corollary 3.10. *Let $f,\ g,\ h\in\mathcal{A}(\mathbb{K})$. Also let the relative index pair of f with respect to h and the index pair of g are $(p-1,m-1)$ and (m,n) respectively where $p,q,m,n\in\mathbb{N}$ such that $p>1$ and $m>1$.Then*

$$\lambda_h^{(p-1,q-1)}(f)\,\tau^{(m,n)}(g)\leq\tau_h^{(p,n)}(f\circ g)\leq\rho_h^{(p-1,q-1)}(f)\,\tau^{(m,n)}(g)\,,$$

and

$$\lambda_h^{(p-1,q-1)}(f)\,\overline{\tau}^{(m,n)}(g)\leq\overline{\tau}_h^{(p,n)}(f\circ g)\leq\rho_h^{(p-1,q-1)}(f)\,\overline{\tau}^{(m,n)}(g)\,.$$

Reasoning similarly as in the proof of the Corollary 3.9 one can easily deduce the conclusion of Corollary 3.10, and so its proof is omitted.

Theorem 3.11. *Let $f,\ g,\ h,\ k\in\mathcal{A}(\mathbb{K})$. Also let the index pairs of $f,\ g,\ h$ and k are (p,q), (m,n), (a,b) and (c,d) respectively where $a,b,c,d,p,q,m,n\in\mathbb{N}$.*

(i) *If either ($q=m$, $a=c=p$, $q\geq n$) or ($q<m$, $c=p$, $a=p+m-q$, $q\geq n$) holds and $\lambda^{(p,q)}(f)>0$, $0<\lambda_h^{(b,n)}(f\circ g)\leq\rho_h^{(b,n)}(f\circ g)<\infty$,*

$0<\lambda_k^{(d,q)}(f)\leq\rho_k^{(d,q)}(f)<\infty$ *then*

$$\begin{aligned}\frac{\lambda_h^{(b,n)}(f\circ g)}{\rho_k^{(d,q)}(f)} &\leq \varliminf_{r\to+\infty}\frac{\log^{[b]}\widehat{|h|}\left(|f\circ g|(r)\right)}{\log^{[d]}\widehat{|k|}\left(|f|\left(\exp^{[q-n]}r\right)\right)}\\ &\leq \min\left\{\frac{\lambda_h^{(b,n)}(f\circ g)}{\lambda_k^{(d,q)}(f)},\frac{\rho_h^{(b,n)}(f\circ g)}{\rho_k^{(d,q)}(f)}\right\}\\ &\leq \max\left\{\frac{\lambda_h^{(b,n)}(f\circ g)}{\lambda_k^{(d,q)}(f)},\frac{\rho_h^{(b,n)}(f\circ g)}{\rho_k^{(d,q)}(f)}\right\}\\ &\leq \varlimsup_{r\to+\infty}\frac{\log^{[b]}\widehat{|h|}(|f\circ g|(r))}{\log^{[d]}\widehat{|k|}\left(|f|(\exp^{[q-n]}r)\right)}\leq\frac{\rho_h^{(b,n)}(f\circ g)}{\lambda_k^{(d,q)}(f)}.\end{aligned}$$

(ii) *If* $q>m,\ a=c=p,\ \lambda^{(p,q)}(f)>0,\ 0<\lambda_h^{(b,q+n-m)}(f\circ g\leq \rho_h^{(b,q+n-m)}(f\circ g)<\infty$ *and* $0<\lambda_k^{(d,q)}(f)\leq\rho_k^{(d,q)}(f)<\infty$ *then*

$$\begin{aligned}\frac{\lambda_h^{(b,q+n-m)}(f\circ g)}{\rho_k^{(d,q)}(f)} &\leq \varliminf_{r\to+\infty}\frac{\log^{[b]}\widehat{|h|}(|f\circ g|(r))}{\log^{[d]}\widehat{|k|}\left(|f|\left(\exp^{[m-n]}r\right)\right)}\\ &\leq \min\left\{\frac{\lambda_h^{(b,q+n-m)}(f\circ g)}{\lambda_k^{(d,q)}(f)},\frac{\rho_h^{(b,q+n-m)}(f\circ g)}{\rho_k^{(d,q)}(f)}\right\}\\ &\leq \max\left\{\frac{\lambda_h^{(b,q+n-m)}(f\circ g)}{\lambda_k^{(d,q)}(f)},\frac{\rho_h^{(b,q+n-m)}(f\circ g)}{\rho_k^{(d,q)}(f)}\right\}\\ &\leq \varlimsup_{r\to+\infty}\frac{\log^{[b]}\widehat{|h|}\left(|f\circ g|(r)\right)}{\log^{[d]}\widehat{|k|}\left(|f|\left(\exp^{[m-n]}r\right)\right)}\\ &\leq \frac{\rho_h^{(b,q+n-m)}(f\circ g)}{\lambda_k^{(d,q)}(f)}.\end{aligned}$$

Proof. Let either ($q=m,\ a=c=p,\ q\geq n$) or ($q<m,\ c=p,\ a=p+m-q,\ q\geq n$) hold and $\lambda^{(p,q)}(f)>0$. Then in view of Theorem 3.4, the index-pair of $f\circ g$ is (p,n) or $(p+m-q,n)$ respectively and therefore by Definition 1.6, $\rho_h^{(b,n)}(f\circ g)$ (respectively $\lambda_h^{(b,n)}(f)$) and $\rho_k^{(d,q)}(f)$ (respectively $\lambda_k^{(d,q)}(f)$) exist.

Now from the definition of $\rho_k^{(d,q)}(f)$ and $\lambda_h^{(b,n)}(f\circ g)$, we have for arbitrary positive ε and for all sufficiently large positive numbers of r

that

$$\log^{[b]} \widehat{|h|} \left(|f \circ g|(r)\right) \left(\lambda_h^{(b,n)}(f \circ g) - \varepsilon\right) \log^{[n]} r \tag{3.23}$$

and

$$\log^{[d]} \widehat{|k|} \left(|f| \left(\exp^{[q-n]} r\right)\right) \leq \left(\rho_k^{(d,q)}(f) + \varepsilon\right) \log^{[n]} r. \tag{3.24}$$

Now from (3.23) and (3.24), it follows for all sufficiently large positive numbers of r that

$$\frac{\log^{[b]} \widehat{|h|} \left(|f \circ g|(r)\right)}{\log^{[d]} \widehat{|k|} \left(|f| \left(\exp^{[q-n]} r\right)\right)} \frac{\left(\lambda_h^{(b,n)}(f \circ g) - \varepsilon\right) \log^{[n]} r}{\left(\rho_k^{(d,q)}(f) + \varepsilon\right) \log^{[n]} r} .$$

As $\varepsilon\,(> 0)$ is arbitrary, we obtain that

$$\varliminf_{r \to +\infty} \frac{\log^{[b]} \widehat{|h|} \left(|f \circ g|(r)\right)}{\log^{[d]} \widehat{|k|} \left(|f| \left(\exp^{[q-n]} r\right)\right)} \frac{\lambda_h^{(b,n)}(f \circ g)}{\rho_k^{(d,q)}(f)} . \tag{3.25}$$

Again we get for a sequence of positive numbers of r tending to infinity that

$$\log^{[b]} \widehat{|h|} \left(|f \circ g|(r)\right) \leq \left(\lambda_h^{(b,n)}(f \circ g) + \varepsilon\right) \log^{[n]} r \tag{3.26}$$

and for all sufficiently large positive numbers of r that

$$\log^{[d]} \widehat{|k|} \left(|f| \left(\exp^{[q-n]} r\right)\right) \left(\lambda_k^{(d,q)}(f) - \varepsilon\right) \log^{[n]} r. \tag{3.27}$$

Combining (3.26) and (3.27), we get for a sequence of positive numbers of r tending to infinity that

$$\frac{\log^{[b]} \widehat{|h|} \left(|f \circ g|(r)\right)}{\log^{[d]} \widehat{|k|} \left(|f| \left(\exp^{[q-n]} r\right)\right)} \leq \frac{\left(\lambda_h^{(b,n)}(f \circ g) + \varepsilon\right) \log^{[n]} r}{\left(\lambda_k^{(d,q)}(f) - \varepsilon\right) \log^{[n]} r} .$$

Since $\varepsilon\,(> 0)$ is arbitrary, it follows that

$$\varliminf_{r \to +\infty} \frac{\log^{[b]} \widehat{|h|} \left(|f \circ g|(r)\right)}{\log^{[d]} \widehat{|k|} \left(|f| \left(\exp^{[q-n]} r\right)\right)} \leq \frac{\lambda_h^{(b,n)}(f \circ g)}{\lambda_k^{(d,q)}(f)} . \tag{3.28}$$

Also for a sequence of positive numbers of r tending to infinity that

$$\log^{[d]} \widehat{|k|} \left(|f| \left(\exp^{[q-n]} r\right)\right) \leq \left(\lambda_k^{(d,q)}(f) + \varepsilon\right) \log^{[n]} r. \tag{3.29}$$

Now from (3.23) and (3.29) , we obtain for a sequence of positive numbers of r tending to infinity that

$$\frac{\log^{[b]} \widehat{|h|} \left(|f \circ g| \left(r \right) \right)}{\log^{[d]} \widehat{|k|} \left(|f| \left(\exp^{[q-n]} r \right) \right)} \geq \frac{\left(\lambda_h^{(b,n)} \left(f \circ g \right) - \varepsilon \right) \log^{[n]} r}{\left(\lambda_k^{(d,q)} \left(f \right) + \varepsilon \right) \log^{[n]} r} .$$

As $\varepsilon \, (> 0)$ is arbitrary, we get from above that

$$\overline{\lim_{r \to +\infty}} \frac{\log^{[b]} \widehat{|h|} \left(|f \circ g| \left(r \right) \right)}{\log^{[d]} \widehat{|k|} \left(|f| \left(\exp^{[q-n]} r \right) \right)} \geq \frac{\lambda_h^{(b,n)} \left(f \circ g \right)}{\lambda_k^{(d,q)} \left(f \right)} . \tag{3.30}$$

Also we obtain for all sufficiently large positive numbers of r that

$$\log^{[b]} \widehat{|h|} \left(|f \circ g| \left(r \right) \right) \leq \left(\rho_h^{(b,n)} \left(f \circ g \right) + \varepsilon \right) \log^{[n]} r. \tag{3.31}$$

Now it follows from (3.27) and (3.31) for all sufficiently large positive numbers of r that

$$\frac{\log^{[b]} \widehat{|h|} \left(|f \circ g| \left(r \right) \right)}{\log^{[d]} \widehat{|k|} \left(|f| \left(\exp^{[q-n]} r \right) \right)} \leq \frac{\left(\rho_h^{(b,n)} \left(f \circ g \right) + \varepsilon \right) \log^{[n]} r}{\left(\lambda_k^{(d,q)} \left(f \right) - \varepsilon \right) \log^{[n]} r}.$$

Since $\varepsilon \, (> 0)$ is arbitrary, we obtain that

$$\overline{\lim_{r \to +\infty}} \frac{\log^{[b]} \widehat{|h|} \left(|f \circ g| \left(r \right) \right)}{\log^{[d]} \widehat{|k|} \left(|f| \left(\exp^{[q-n]} r \right) \right)} \leq \frac{\rho_h^{(b,n)} \left(f \circ g \right)}{\lambda_k^{(d,q)} \left(f \right)}. \tag{3.32}$$

Further from the definition of $\rho_k^{(d,q)} \left(f \right)$, we get for a sequence of positive numbers of r tending to infinity that

$$\log^{[d]} \widehat{|k|} \left(|f| \left(\exp^{[q-n]} r \right) \right) \left(\rho_k^{(d,q)} \left(f \right) - \varepsilon \right) \log^{[n]} r. \tag{3.33}$$

Now from (3.31) and (3.33) , it follows for a sequence of positive numbers of r tending to infinity that

$$\frac{\log^{[b]} \widehat{|h|} \left(|f \circ g| \left(r \right) \right)}{\log^{[d]} \widehat{|k|} \left(|f| \left(\exp^{[q-n]} r \right) \right)} \leq \frac{\left(\rho_h^{(b,n)} \left(f \circ g \right) + \varepsilon \right) \log^{[n]} r}{\left(\rho_k^{(d,q)} \left(f \right) - \varepsilon \right) \log^{[n]} r}.$$

As $\varepsilon \, (> 0)$ is arbitrary, we obtain that

$$\underline{\lim_{r \to +\infty}} \frac{\log^{[b]} \widehat{|h|} \left(|f \circ g| \left(r \right) \right)}{\log^{[d]} \widehat{|k|} \left(|f| \left(\exp^{[q-n]} r \right) \right)} \leq \frac{\rho_h^{(b,n)} \left(f \circ g \right)}{\rho_k^{(d,q)} \left(f \right)}. \tag{3.34}$$

Again we obtain for a sequence of positive numbers of r tending to infinity that

$$\log^{[b]} \widehat{|h|}\left(|f \circ g|\left(r\right)\right) \left(\rho_h^{(b,n)}\left(f \circ g\right) - \varepsilon\right) \log^{[n]} r. \tag{3.35}$$

So combining (3.24) and (3.35), we get for a sequence of positive numbers of r tending to infinity that

$$\frac{\log^{[b]} \widehat{|h|}\left(|f \circ g|\left(r\right)\right)}{\log^{[d]} \widehat{|k|}\left(|f|\left(\exp^{[q-n]} r\right)\right)} \frac{\left(\rho_h^{(b,n)}\left(f \circ g\right) - \varepsilon\right) \log^{[n]} r}{\left(\rho_k^{(d,q)}\left(f\right) + \varepsilon\right) \log^{[n]} r} .$$

Since $\varepsilon\,(> 0)$ is arbitrary, it follows that

$$\overline{\lim_{r \to +\infty}} \frac{\log^{[b]} \widehat{|h|}\left(|f \circ g|\left(r\right)\right)}{\log^{[d]} \widehat{|k|}\left(|f|\left(\exp^{[q-n]} r\right)\right)} \frac{\rho_h^{(b,n)}\left(f \circ g\right)}{\rho_k^{(d,q)}\left(f\right)}. \tag{3.36}$$

Thus the first part of the theorem follows from (3.25), (3.28), (3.30), (3.32), (3.34) and (3.36).

Analogously, the second part of the theorem can be derived in a like manner. □

The following theorem can be proved in the line of Theorem 3.11 and so its proof is omitted.

Theorem 3.12. *Let f, g, h, $l \in \mathcal{A}(\mathbb{K})$. Also let the index pairs of f, g, h and l are (p,q), (m,n), (a,b) and (x,y) respectively where $a,b,p,q,m,n,x,y \in \mathbb{N}$.*

(i) *If either $(q = m = x, a = p)$ or $(q < m = x, a = p + m - q)$ holds, $\lambda^{(m,n)}(g) > 0$, $0 < \lambda_h^{(b,n)}(f \circ g) \le \rho_h^{(b,n)}(f \circ g) < \infty$, $0 < \lambda_l^{(y,n)}(g) \le \rho_l^{(y,n)}(g) < \infty$ then*

$$\begin{aligned} \frac{\lambda_h^{(b,n)}(f \circ g)}{\rho_l^{(y,n)}(g)} &\le \underline{\lim}_{r \to +\infty} \frac{\log^{[b]} \widehat{|h|}\left(|f \circ g|(r)\right)}{\log^{[y]} \widehat{|l|}\left(|g|(r)\right)} \\ &\le \min\left\{\frac{\lambda_h^{(b,n)}(f \circ g)}{\lambda_l^{(y,n)}(g)}, \frac{\rho_h^{(b,n)}(f \circ g)}{\rho_l^{(y,n)}(g)}\right\} \\ &\le \max\left\{\frac{\lambda_h^{(b,n)}(f \circ g)}{\lambda_l^{(y,n)}(g)}, \frac{\rho_h^{(b,n)}(f \circ g)}{\rho_l^{(y,n)}(g)}\right\} \end{aligned}$$

$$\leq \varlimsup_{r\to+\infty} \frac{\log^{[b]} \widehat{|h|}\left(|f\circ g|(r)\right)}{\log^{[y]} \widehat{|l|}\left(|g|(r)\right)}$$

$$\leq \frac{\rho_h^{(b,n)}(f\circ g)}{\lambda_l^{(y,n)}(g)}.$$

(ii) *If* $q > m = x, a = p, \lambda^{(m,n)}(g) > 0$, $0 < \lambda_h^{(b,q+n-m)}(f\circ g) \leq \rho_h^{(b,q+n-m)}(f\circ g) < \infty$, $0 < \lambda_l^{(y,n)}(g) \leq \rho_l^{(y,n)}(g) < \infty$ *then*

$$\frac{\lambda_h^{(b,q+n-m)}(f\circ g)}{\rho_l^{(y,n)}(g)} \leq \varliminf_{r\to+\infty} \frac{\log^{[b]} \widehat{|h|}\left(|f\circ g|\left(\exp^{[q-m]} r\right)\right)}{\log^{[y]} \widehat{|l|}\left(|g|(r)\right)}$$

$$\leq \min\left\{\frac{\lambda_h^{(b,q+n-m)}(f\circ g)}{\lambda_l^{(y,n)}(g)}, \frac{\rho_h^{(b,q+n-m)}(f\circ g)}{\rho_l^{(y,n)}(g)}\right\}$$

$$\leq \max\left\{\frac{\lambda_h^{(b,q+n-m)}(f\circ g)}{\lambda_l^{(y,n)}(g)}, \frac{\rho_h^{(b,q+n-m)}(f\circ g)}{\rho_l^{(y,n)}(g)}\right\}$$

$$\leq \varlimsup_{r\to+\infty} \frac{\log^{[b]} \widehat{|h|}\left(|f\circ g|\left(\exp^{[q-m]} r\right)\right)}{\log^{[y]} \widehat{|l|}\left(|g|(r)\right)}$$

$$\leq \frac{\rho_h^{(b,q+n-m)}(f\circ g)}{\lambda_l^{(y,n)}(g)}.$$

The proofs of the following four theorems can be carried out as of the Theorem 3.11, therefore we omit the details.

THEOREM 3.13. *Let* $f, g, h, k \in \mathcal{A}(\mathbb{K})$. *Also let the index pairs of* f, g, h *and* k *are* (p,q), (m,n), (a,b) *and* (c,d) *respectively where* $a,b,c,d,p,q,m,n \in \mathbb{N}$.

(i) *If either* $(q = m, a = c = p, q \geq n)$ *or* $(q < m, c = p, a = p + m - q, q \geq n)$ *holds,* $0 < \overline{\sigma}_h^{(b,n)}(f\circ g) \leq \sigma_h^{(b,n)}(f\circ g) < \infty$, $0 < \overline{\sigma}_k^{(d,q)}(f) \leq \sigma_k^{(d,q)}(f) < \infty$ *and* $\rho_h^{(b,n)}(f\circ g) = \rho_k^{(d,q)}(f)$, *then*

$$\frac{\overline{\sigma}_h^{(b,n)}(f\circ g)}{\sigma_k^{(d,q)}(f)} \leq \varliminf_{r\to+\infty} \frac{\log^{[b-1]} \widehat{|h|}\left(|f\circ g|(r)\right)}{\log^{[d-1]} \widehat{|k|}\left(|f|\left(\exp^{[q-n]} r\right)\right)}$$

$$\leq \min\left\{\frac{\overline{\sigma}_h^{(b,n)}(f\circ g)}{\overline{\sigma}_k^{(d,q)}(f)}, \frac{\sigma_h^{(b,n)}(f\circ g)}{\sigma_k^{(d,q)}(f)}\right\}$$

$$\leq \max\left\{\frac{\overline{\sigma}_h^{(b,n)}(f\circ g)}{\overline{\sigma}_k^{(d,q)}(f)}, \frac{\sigma_h^{(b,n)}(f\circ g)}{\sigma_k^{(d,q)}(f)}\right\}$$

$$\leq \varlimsup_{r\to+\infty} \frac{\log^{[b-1]}\widehat{|h|}\left(|f\circ g|(r)\right)}{\log^{[d-1]}\widehat{|k|}\left(|f|\left(\exp^{[q-n]} r\right)\right)}$$

$$\leq \frac{\sigma_h^{(b,n)}(f\circ g)}{\overline{\sigma}_k^{(d,q)}(f)}.$$

(ii) *If* $q > m, a = c = p, 0 < \overline{\sigma}_h^{(b,q+n-m)}(f\circ g) \leq \sigma_h^{(b,q+n-m)}(f\circ g) < \infty$, $0 < \overline{\sigma}_k^{(d,q)}(f) \leq \sigma_k^{(d,q)}(f) < \infty$ *and* $\rho_h^{(b,q+n-m)}(f\circ g) = \rho_k^{(d,q)}(f)$, *then*

$$\frac{\overline{\sigma}_h^{(b,q+n-m)}(f\circ g)}{\sigma_k^{(d,q)}(f)} \leq \varliminf_{r\to+\infty} \frac{\log^{[b-1]}\widehat{|h|}\left(|f\circ g|(r)\right)}{\log^{[d-1]}\widehat{|k|}\left(|f|\left(\exp^{[m-n]} r\right)\right)}$$

$$\leq \min\left\{\frac{\overline{\sigma}_h^{(b,q+n-m)}(f\circ g)}{\overline{\sigma}_k^{(d,q)}(f)}, \frac{\sigma_h^{(b,q+n-m)}(f\circ g)}{\sigma_k^{(d,q)}(f)}\right\}$$

$$\leq \max\left\{\frac{\overline{\sigma}_h^{(b,q+n-m)}(f\circ g)}{\overline{\sigma}_k^{(d,q)}(f)}, \frac{\sigma_h^{(b,q+n-m)}(f\circ g)}{\sigma_k^{(d,q)}(f)}\right\}$$

$$\leq \varlimsup_{r\to+\infty} \frac{\log^{[b-1]}\widehat{|h|}\left(|f\circ g|(r)\right)}{\log^{[d-1]}\widehat{|k|}\left(|f|\left(\exp^{[m-n]} r\right)\right)}$$

$$\leq \frac{\sigma_h^{(b,q+n-m)}(f\circ g)}{\overline{\sigma}_k^{(d,q)}(f)}.$$

THEOREM 3.14. *Let* $f, g, h, k \in \mathcal{A}(\mathbb{K})$. *Also let the index pairs of* f, g, h *and* k *are* (p,q), (m,n), (a,b) *and* (c,d) *respectively where* $a,b,c,d,p,q,m,n \in \mathbb{N}$.

(i) *If either* ($q = m, a = c = p, q \geq n$) *or* ($q < m, c = p, a = p + m - q, q \geq n$) *holds,* $0 < \tau_h^{(b,n)}(f\circ g) \leq \overline{\tau}_h^{(b,n)}(f\circ g) < \infty, 0 < \tau_k^{(d,q)}(f) \leq \overline{\tau}_k^{(d,q)}(f) < \infty$ *and* $\lambda_h^{(b,n)}(f\circ g) = \lambda_k^{(d,q)}(f)$, *then*

$$\frac{\tau_h^{(b,n)}(f\circ g)}{\overline{\tau}_k^{(d,q)}(f)} \leq \varliminf_{r\to+\infty} \frac{\log^{[b-1]}\widehat{|h|}\left(|f\circ g|(r)\right)}{\log^{[d-1]}\widehat{|k|}\left(|f|\left(\exp^{[q-n]} r\right)\right)}$$

$$\leq \min\left\{\frac{\tau_h^{(b,n)}(f\circ g)}{\tau_k^{(d,q)}(f)}, \frac{\overline{\tau}_h^{(b,n)}(f\circ g)}{\overline{\tau}_k^{(d,q)}(f)}\right\}$$

$$\leq \max\left\{\frac{\tau_h^{(b,n)}(f\circ g)}{\tau_k^{(d,q)}(f)}, \frac{\overline{\tau}_h^{(b,n)}(f\circ g)}{\overline{\tau}_k^{(d,q)}(f)}\right\}$$

$$\leq \varlimsup_{r\to+\infty} \frac{\log^{[b-1]} \widehat{|h|} (|f \circ g|(r))}{\log^{[d-1]} \widehat{|k|} \left(|f| \left(\exp^{[q-n]} r\right)\right)}$$

$$\leq \frac{\overline{\tau}_h^{(b,n)} (f \circ g)}{\tau_k^{(d,q)} (f)}.$$

(ii) *If $q > m, a = c = p$, $0 < \tau_h^{(b,q+n-m)} (f \circ g) \leq \overline{\tau}_h^{(b,q+n-m)} (f \circ g) < \infty$, $0 < \tau_k^{(d,q)} (f) \leq \overline{\tau}_k^{(d,q)} (f) < \infty$ and $\lambda_h^{(b,q+n-m)} (f \circ g) = \lambda_k^{(d,q)} (f)$, then*

$$\frac{\tau_h^{(b,q+n-m)} (f \circ g)}{\overline{\tau}_k^{(d,q)} (f)} \leq \varliminf_{r\to+\infty} \frac{\log^{[b-1]} \widehat{|h|} (|f \circ g| (r))}{\log^{[d-1]} \widehat{|k|} \left(|f| \left(\exp^{[m-n]} r\right)\right)}$$

$$\leq \min \left\{ \frac{\tau_h^{(b,q+n-m)} (f \circ g)}{\tau_k^{(d,q)} (f)}, \frac{\overline{\tau}_h^{(b,q+n-m)} (f \circ g)}{\overline{\tau}_k^{(d,q)} (f)} \right\}$$

$$\leq \max \left\{ \frac{\tau_h^{(b,q+n-m)}(f \circ g)}{\tau_k^{(d,q)}(f)}, \frac{\overline{\tau}_h^{(b,q+n-m)}(f \circ g)}{\overline{\tau}_k^{(d,q)}(f)} \right\}$$

$$\leq \varlimsup_{r\to+\infty} \frac{\log^{[b-1]} \widehat{|h|} (|f \circ g| (r))}{\log^{[d-1]} \widehat{|k|} \left(|f| \left(\exp^{[m-n]} r\right)\right)}$$

$$\leq \frac{\overline{\tau}_h^{(b,q+n-m)} (f \circ g)}{\tau_k^{(d,q)} (f)}.$$

THEOREM 3.15. *Let f, g, h, $k \in \mathcal{A}(\mathbb{K})$. Also let the index pairs of f, g, h and k are (p,q), (m,n), (a,b) and (c,d) respectively where $a,b,c,d,p,q,m,n \in \mathbb{N}$.*

(i) *If either $(q = m, a = c = p, q \geq n)$ or $(q < m, c = p, a = p + m - q, q \geq n)$ holds, $0 < \overline{\sigma}_h^{(b,n)} (f \circ g) \leq \sigma_h^{(b,n)} (f \circ g) < \infty$,0¡$\tau_k^{(d,q)} (f) \leq \overline{\tau}_k^{(d,q)} (f) < \infty$ and $\rho_h^{(b,n)} (f \circ g) = \lambda_k^{(d,q)} (f)$, then*

$$\frac{\overline{\sigma}_h^{(b,n)} (f \circ g)}{\overline{\tau}_k^{(d,q)}(f)} \leq \varliminf_{r\to+\infty} \frac{\log^{[b-1]} \widehat{|h|} (|f \circ g| (r))}{\log^{[d-1]} \widehat{|k|} \left(|f| \left(\exp^{[q-n]} r\right)\right)}$$

$$\leq \min \left\{ \frac{\overline{\sigma}_h^{(b,n)} (f \circ g)}{\tau_k^{(d,q)} (f)}, \frac{\sigma_h^{(b,n)} (f \circ g)}{\overline{\tau}_k^{(d,q)} (f)} \right\}$$

$$\leq \max \left\{ \frac{\overline{\sigma}_h^{(b,n)} (f \circ g)}{\tau_k^{(d,q)} (f)}, \frac{\sigma_h^{(b,n)} (f \circ g)}{\overline{\tau}_k^{(d,q)} (f)} \right\}$$

$$\leq \overline{\lim_{r\to+\infty}} \frac{\log^{[b-1]} \widehat{|h|}\left(|f\circ g|(r)\right)}{\log^{[d-1]} \widehat{|k|}\left(|f|\left(\exp^{[q-n]} r\right)\right)}$$
$$\leq \frac{\sigma_h^{(b,n)}(f\circ g)}{\tau_k^{(d,q)}(f)}.$$

(ii) *If $q > m, a = c = p, 0 < \overline{\sigma}_h^{(b,q+n-m)}(f\circ g) \leq \sigma_h^{(b,q+n-m)}(f\circ g) < \infty, 0 < \tau_k^{(d,q)}(f) \leq \overline{\tau}_k^{(d,q)}(f) < \infty$ and $\rho_h^{(b,q+n-m)}(f\circ g) = \lambda_k^{(d,q)}(f)$, then*

$$\frac{\overline{\sigma}_h^{(b,q+n-m)}(f\circ g)}{\overline{\tau}_k^{(d,q)}(f)} \leq \underline{\lim}_{r\to+\infty} \frac{\log^{[b-1]} \widehat{|h|}\left(|f\circ g|(r)\right)}{\log^{[d-1]} \widehat{|k|}\left(|f|\left(\exp^{[m-n]} r\right)\right)}$$
$$\leq \min\left\{\frac{\overline{\sigma}_h^{(b,q+n-m)}(f\circ g)}{\tau_k^{(d,q)}(f)}, \frac{\sigma_h^{(b,q+n-m)}(f\circ g)}{\overline{\tau}_k^{(d,q)}(f)}\right\}$$
$$\leq \max\left\{\frac{\overline{\sigma}_h^{(b,q+n-m)}(f\circ g)}{\tau_k^{(d,q)}(f)}, \frac{\sigma_h^{(b,q+n-m)}(f\circ g)}{\overline{\tau}_k^{(d,q)}(f)}\right\}$$
$$\leq \overline{\lim_{r\to+\infty}} \frac{\log^{[b-1]} \widehat{|h|}\left(|f\circ g|(r)\right)}{\log^{[d-1]} \widehat{|k|}\left(|f|\left(\exp^{[m-n]} r\right)\right)}$$
$$\leq \frac{\sigma_h^{(b,q+n-m)}(f\circ g)}{\tau_k^{(d,q)}(f)}.$$

THEOREM 3.16. *Let $f,\ g,\ h,\ k \in \mathcal{A}(\mathbb{K})$. Also let the index pairs of $f,\ g,\ h$ and k are (p,q), (m,n), (a,b) and (c,d) respectively where $a,b,c,d,p,q,m,n \in \mathbb{N}$.*

(i) *If either $(q = m, a = c = p, q \geq n)$ or $(q < m, c = p, a = p + m - q, q \geq n)$ holds, $0 < \tau_h^{(b,n)}(f\circ g) \leq \overline{\tau}_h^{(b,n)}(f\circ g) < \infty, 0 < \overline{\sigma}_k^{(d,q)}(f) \leq \sigma_k^{(d,q)}(f) < \infty$ and $\lambda_h^{(b,n)}(f\circ g) = \rho_k^{(d,q)}(f)$, then*

$$\frac{\tau_h^{(b,n)}(f\circ g)}{\sigma_k^{(d,q)}(f)} \leq \underline{\lim}_{r\to+\infty} \frac{\log^{[b-1]} \widehat{|h|}\left(|f\circ g|(r)\right)}{\log^{[d-1]} \widehat{|k|}\left(|f|\left(\exp^{[q-n]} r\right)\right)}$$
$$\leq \min\left\{\frac{\tau_h^{(b,n)}(f\circ g)}{\overline{\sigma}_k^{(d,q)}(f)}, \frac{\overline{\tau}_h^{(b,n)}(f\circ g)}{\sigma_k^{(d,q)}(f)}\right\}$$
$$\leq \max\left\{\frac{\tau_h^{(b,n)}(f\circ g)}{\overline{\sigma}_k^{(d,q)}(f)}, \frac{\overline{\tau}_h^{(b,n)}(f\circ g)}{\sigma_k^{(d,q)}(f)}\right\}$$

$$\leq \varlimsup_{r\to+\infty} \frac{\log^{[b-1]} \widehat{|h|}\left(|f\circ g|\left(r\right)\right)}{\log^{[d-1]} \widehat{|k|}\left(|f|\left(\exp^{[q-n]} r\right)\right)}$$

$$\leq \frac{\overline{\tau}_h^{(b,n)}\left(f\circ g\right)}{\overline{\sigma}_k^{(d,q)}\left(f\right)}.$$

(ii) *If* $q > m, a = c = p, 0 < \tau_h^{(b,q+n-m)}\left(f\circ g\right) \leq \overline{\tau}_h^{(b,q+n-m)}\left(f\circ g\right) < \infty, 0 < \overline{\sigma}_k^{(d,q)}\left(f\right) \leq \sigma_k^{(d,q)}\left(f\right) < \infty$ and $\lambda_h^{(b,q+n-m)}\left(f\circ g\right) = \rho_k^{(d,q)}\left(f\right),$ *then*

$$\frac{\tau_h^{(b,q+n-m)}\left(f\circ g\right)}{\sigma_k^{(d,q)}\left(f\right)} \leq \varliminf_{r\to+\infty} \frac{\log^{[b-1]} \widehat{|h|}\left(|f\circ g|\left(r\right)\right)}{\log^{[d-1]} \widehat{|k|}\left(|f|\left(\exp^{[m-n]} r\right)\right)}$$

$$\leq \min\left\{\frac{\tau_h^{(b,q+n-m)}\left(f\circ g\right)}{\overline{\sigma}_k^{(d,q)}\left(f\right)}, \frac{\overline{\tau}_h^{(b,q+n-m)}\left(f\circ g\right)}{\sigma_k^{(d,q)}\left(f\right)}\right\}$$

$$\leq \max\left\{\frac{\tau_h^{(b,q+n-m)}\left(f\circ g\right)}{\overline{\sigma}_k^{(d,q)}\left(f\right)}, \frac{\overline{\tau}_h^{(b,q+n-m)}\left(f\circ g\right)}{\sigma_k^{(d,q)}\left(f\right)}\right\}$$

$$\leq \varlimsup_{r\to+\infty} \frac{\log^{[b-1]} \widehat{|h|}\left(|f\circ g|\left(r\right)\right)}{\log^{[d-1]} \widehat{|k|}\left(|f|\left(\exp^{[m-n]} r\right)\right)}$$

$$\leq \frac{\overline{\tau}_h^{(b,q+n-m)}\left(f\circ g\right)}{\overline{\sigma}_k^{(d,q)}\left(f\right)}.$$

Analogously one may formulate the following four theorems without their proofs.

THEOREM 3.17. *Let* $f,\ g,\ h,\ l \in \mathcal{A}\left(\mathbb{K}\right)$. *Also let the index pairs of* $f,\ g,\ h$ *and* l *are* $\left(p,q\right), \left(m,n\right),\ \left(a,b\right)$ *and* $\left(x,y\right)$ *respectively where* $a,b,p,q,m,n,x,y \in \mathbb{N}$.

(i) *If either* $\left(q = m = x, a = p\right)$ *or* $\left(q < m = x, a = p+m-q\right)$ *holds,* $0 < \overline{\sigma}_h^{(b,n)}\left(f\circ g\right) \leq \sigma_h^{(b,n)}\left(f\circ g\right) < \infty$, $0 < \overline{\sigma}_l^{(y,n)}\left(g\right) \leq \sigma_l^{(y,n)}\left(g\right) < \infty$ *and* $\rho_h^{(b,n)}\left(f\circ g\right) = \rho_l^{(y,n)}\left(g\right)$ *then*

$$\frac{\overline{\sigma}_h^{(b,n)}\left(f\circ g\right)}{\sigma_l^{(y,n)}\left(g\right)} \leq \frac{\overline{\sigma}_h^{(b,n)}\left(f\circ g\right)}{\sigma_l^{(y,n)}\left(g\right)}$$

$$\leq \varliminf_{r\to+\infty} \frac{\log^{[b-1]} \widehat{|h|}\left(|f\circ g|\left(r\right)\right)}{\log^{[y-1]} \widehat{|l|}\left(|g|\left(r\right)\right)}$$

$$\leq \min\left\{\frac{\overline{\sigma}_h^{(b,n)}\left(f\circ g\right)}{\overline{\sigma}_l^{(y,n)}\left(g\right)}, \frac{\sigma_h^{(b,n)}\left(f\circ g\right)}{\sigma_l^{(y,n)}\left(g\right)}\right\}$$

$$
\begin{aligned}
&\leq \max\left\{\frac{\overline{\sigma}_h^{(b,n)}(f\circ g)}{\overline{\sigma}_l^{(y,n)}(g)}, \frac{\sigma_h^{(b,n)}(f\circ g)}{\sigma_l^{(y,n)}(g)}\right\}\\
&\leq \varlimsup_{r\to+\infty}\frac{\log^{[b-1]}\widehat{|h|}(|f\circ g|(r))}{\log^{[y-1]}\widehat{|l|}(|g|(r))}\\
&\leq \frac{\sigma_h^{(b,n)}(f\circ g)}{\overline{\sigma}_l^{(y,n)}(g)}.
\end{aligned}
$$

(ii) *If* $q > m = x, a = p, 0 < \overline{\sigma}_h^{(b,q+n-m)}(f\circ g) \leq \sigma_h^{(b,q+n-m)}(f\circ g) < \infty, 0 < \overline{\sigma}_l^{(y,n)}(g) \leq \sigma_l^{(y,n)}(g) < \infty$ and $\rho_h^{(b,q+n-m)}(f\circ g) = \rho_l^{(y,n)}(g)$ *then*

$$
\begin{aligned}
\frac{\overline{\sigma}_h^{(b,q+n-m)}(f\circ g)}{\sigma_l^{(y,n)}(g)} &\leq \varliminf_{r\to+\infty}\frac{\log^{[b-1]}\widehat{|h|}\left(|f\circ g|\left(\exp^{[q-m]}r\right)\right)}{\log^{[y-1]}\widehat{|l|}(|g|(r))}\\
&\leq \min\left\{\frac{\overline{\sigma}_h^{(b,q+n-m)}(f\circ g)}{\overline{\sigma}_l^{(y,n)}(g)}, \frac{\sigma_h^{(b,q+n-m)}(f\circ g)}{\sigma_l^{(y,n)}(g)}\right\}\\
&\leq \max\left\{\frac{\overline{\sigma}_h^{(b,q+n-m)}(f\circ g)}{\overline{\sigma}_l^{(y,n)}(g)}, \frac{\sigma_h^{(b,q+n-m)}(f\circ g)}{\sigma_l^{(y,n)}(g)}\right\}\\
&\leq \varlimsup_{r\to+\infty}\frac{\log^{[b-1]}\widehat{|h|}\left(|f\circ g|\left(\exp^{[q-m]}r\right)\right)}{\log^{[y-1]}\widehat{|l|}(|g|(r))}\\
&\leq \frac{\sigma_h^{(b,q+n-m)}(f\circ g)}{\overline{\sigma}_l^{(y,n)}(g)}.
\end{aligned}
$$

THEOREM 3.18. *Let* $f, g, h, l \in \mathcal{A}(\mathbb{K})$. *Also let the index pairs of* f, g, h *and* l *are* (p,q), (m,n), (a,b) *and* (x,y) *respectively where* $a,b,p,q,m,n,x,y \in \mathbb{N}$.

(i) *If either* $(q = m = x, a = p)$ *or* $(q < m = x, a = p + m - q)$ *holds,* $0 < \tau_h^{(b,n)}(f\circ g) \leq \overline{\tau}_h^{(b,n)}(f\circ g) < \infty, 0 < \tau_l^{(y,n)}(g) \leq \overline{\tau}_l^{(y,n)}(g) < \infty$ *and* $\lambda_h^{(b,n)}(f\circ g) = \lambda_l^{(y,n)}(g)$ *then*

$$
\begin{aligned}
\frac{\tau_h^{(b,n)}(f\circ g)}{\overline{\tau}_l^{(y,n)}(g)} &\leq \varliminf_{r\to+\infty}\frac{\log^{[b-1]}\widehat{|h|}(|f\circ g|(r))}{\log^{[y-1]}\widehat{|l|}(|g|(r))}\\
&\leq \min\left\{\frac{\tau_h^{(b,n)}(f\circ g)}{\tau_l^{(y,n)}(g)}, \frac{\overline{\tau}_h^{(b,n)}(f\circ g)}{\overline{\tau}_l^{(y,n)}(g)}\right\}
\end{aligned}
$$

$$\leq \max\left\{\frac{\tau_h^{(b,n)}(f\circ g)}{\tau_l^{(y,n)}(g)}, \frac{\overline{\tau}_h^{(b,n)}(f\circ g)}{\overline{\tau}_l^{(y,n)}(g)}\right\}$$

$$\leq \varlimsup_{r\to+\infty}\frac{\log^{[b-1]}\widehat{|h|}(|f\circ g|(r))}{\log^{[y-1]}\widehat{|l|}(|g|(r))} \leq \frac{\overline{\tau}_h^{(b,n)}(f\circ g)}{\tau_l^{(y,n)}(g)}.$$

(ii) *If* $q > m = x, a = p, 0 < \tau_h^{(b,q+n-m)}(f\circ g) \leq \overline{\tau}_h^{(b,q+n-m)}(f\circ g) < \infty, 0 < \tau_l^{(y,n)}(g) \leq \overline{\tau}_l^{(y,n)}(g) < \infty$ *and* $\lambda_h^{(b,q+n-m)}(f\circ g) = \lambda_l^{(y,n)}(g)$ *then*

$$\frac{\tau_h^{(b,q+n-m)}(f\circ g)}{\overline{\tau}_l^{(y,n)}(g)} \leq \varliminf_{r\to+\infty}\frac{\log^{[b-1]}\widehat{|h|}\left(|f\circ g|\left(\exp^{[q-m]}r\right)\right)}{\log^{[y-1]}\widehat{|l|}(|g|(r))}$$

$$\leq \min\left\{\frac{\tau_h^{(b,q+n-m)}(f\circ g)}{\tau_l^{(y,n)}(g)}, \frac{\overline{\tau}_h^{(b,q+n-m)}(f\circ g)}{\overline{\tau}_l^{(y,n)}(g)}\right\}$$

$$\leq \max\left\{\frac{\tau_h^{(b,q+n-m)}(f\circ g)}{\tau_l^{(y,n)}(g)}, \frac{\overline{\tau}_h^{(b,q+n-m)}(f\circ g)}{\overline{\tau}_l^{(y,n)}(g)}\right\}$$

$$\leq \varlimsup_{r\to+\infty}\frac{\log^{[b-1]}\widehat{|h|}\left(|f\circ g|\left(\exp^{[q-m]}r\right)\right)}{\log^{[y-1]}\widehat{|l|}(|g|(r))}$$

$$\leq \frac{\overline{\tau}_h^{(b,q+n-m)}(f\circ g)}{\tau_l^{(y,n)}(g)}.$$

THEOREM 3.19. *Let* $f, g, h, l \in \mathcal{A}(\mathbb{K})$. *Also let the index pairs of* f, g, h *and* l *are* (p,q), (m,n), (a,b) *and* (x,y) *respectively where* $a,b,p,q,m,n,x,y \in \mathbb{N}$.

(i) *If either* $(q = m = x, a = p)$ *or* $(q < m = x, a = p+m-q)$ *holds,* $0 < \overline{\sigma}_h^{(b,n)}(f\circ g) \leq \sigma_h^{(b,n)}(f\circ g) < \infty, 0 < \tau_l^{(y,n)}(g) \leq \overline{\tau}_l^{(y,n)}(g) < \infty$ *and* $\rho_h^{(b,n)}(f\circ g) = \lambda_l^{(y,n)}(g)$ *then*

$$\frac{\overline{\sigma}_h^{(b,n)}(f\circ g)}{\overline{\tau}_l^{(y,n)}(g)} \leq \varliminf_{r\to+\infty}\frac{\log^{[b-1]}\widehat{|h|}(|f\circ g|(r))}{\log^{[y-1]}\widehat{|l|}(|g|(r))}$$

$$\leq \min\left\{\frac{\overline{\sigma}_h^{(b,n)}(f\circ g)}{\tau_l^{(y,n)}(g)}, \frac{\sigma_h^{(b,n)}(f\circ g)}{\overline{\tau}_l^{(y,n)}(g)}\right\}$$

$$\leq \max\left\{\frac{\overline{\sigma}_h^{(b,n)}(f\circ g)}{\tau_l^{(y,n)}(g)}, \frac{\sigma_h^{(b,n)}(f\circ g)}{\overline{\tau}_l^{(y,n)}(g)}\right\}$$

$$\leq \overline{\lim_{r\to+\infty}} \frac{\log^{[b-1]} \widehat{|h|}\left(|f\circ g|(r)\right)}{\log^{[y-1]} \widehat{|l|}\left(|g|(r)\right)}$$
$$\leq \frac{\sigma_h^{(b,n)}(f\circ g)}{\tau_l^{(y,n)}(g)}.$$

(ii) *If* $q > m = x, a = p, 0 < \overline{\sigma}_h^{(b,q+n-m)}(f\circ g) \leq \sigma_h^{(b,q+n-m)}(f\circ g) < \infty, 0 < \tau_l^{(y,n)}(g) \leq \overline{\tau}_l^{(y,n)}(g) < \infty$ and $\rho_h^{(b,q+n-m)}(f\circ g) = \lambda_l^{(y,n)}(g)$ *then*

$$\frac{\overline{\sigma}_h^{(b,q+n-m)}(f\circ g)}{\overline{\tau}_l^{(y,n)}(g)} \leq \underline{\lim}_{r\to+\infty} \frac{\log^{[b-1]} \widehat{|h|}\left(|f\circ g|\left(\exp^{[q-m]} r\right)\right)}{\log^{[y-1]} \widehat{|l|}\left(|g|(r)\right)}$$
$$\leq \min\left\{\frac{\overline{\sigma}_h^{(b,q+n-m)}(f\circ g)}{\tau_l^{(y,n)}(g)}, \frac{\sigma_h^{(b,q+n-m)}(f\circ g)}{\overline{\tau}_l^{(y,n)}(g)}\right\}$$
$$\leq \max\left\{\frac{\overline{\sigma}_h^{(b,q+n-m)}(f\circ g)}{\tau_l^{(y,n)}(g)}, \frac{\sigma_h^{(b,q+n-m)}(f\circ g)}{\overline{\tau}_l^{(y,n)}(g)}\right\}$$
$$\leq \overline{\lim_{r\to+\infty}} \frac{\log^{[b-1]} \widehat{|h|}\left(|f\circ g|\left(\exp^{[q-m]} r\right)\right)}{\log^{[y-1]} \widehat{|l|}\left(|g|(r)\right)}$$
$$\leq \frac{\sigma_h^{(b,q+n-m)}(f\circ g)}{\tau_l^{(y,n)}(g)}.$$

THEOREM 3.20. *Let* $f, g, h, l \in \mathcal{A}(\mathbb{K})$. *Also let the index pairs of* f, g, h *and* l *are* $(p,q), (m,n), (a,b)$ *and* (x,y) *respectively where* $a, b, p, q, m, n, x, y \in \mathbb{N}$.

(i) *If either* $(q = m = x, a = p)$ *or* $(q < m = x, a = p+m-q)$ *holds,* $0 < \tau_h^{(b,n)}(f\circ g) \leq \overline{\tau}_h^{(b,n)}(f\circ g) < \infty, 0 < \overline{\sigma}_l^{(y,n)}(g) \leq \sigma_l^{(y,n)}(g) < \infty$ *and* $\lambda_h^{(b,n)}(f\circ g) = \rho_l^{(y,n)}(g)$ *then*

$$\frac{\tau_h^{(b,n)}(f\circ g)}{\sigma_l^{(y,n)}(g)} \leq \underline{\lim}_{r\to+\infty} \frac{\log^{[b-1]} \widehat{|h|}\left(|f\circ g|(r)\right)}{\log^{[y-1]} \widehat{|l|}\left(|g|(r)\right)}$$
$$\leq \min\left\{\frac{\tau_h^{(b,n)}(f\circ g)}{\overline{\sigma}_l^{(y,n)}(g)}, \frac{\overline{\tau}_h^{(b,n)}(f\circ g)}{\sigma_l^{(y,n)}(g)}\right\}$$
$$\leq \max\left\{\frac{\tau_h^{(b,n)}(f\circ g)}{\overline{\sigma}_l^{(y,n)}(g)}, \frac{\overline{\tau}_h^{(b,n)}(f\circ g)}{\sigma_l^{(y,n)}(g)}\right\}$$

$$\leq \varlimsup_{r\to+\infty} \frac{\log^{[b-1]} \widehat{|h|}\left(|f\circ g|(r)\right)}{\log^{[y-1]} \widehat{|l|}\left(|g|(r)\right)}$$
$$\leq \frac{\overline{\tau}_h^{(b,n)}(f\circ g)}{\overline{\sigma}_l^{(y,n)}(g)}.$$

(ii) *If* $q > m = x, a = p, 0 < \tau_h^{(b,q+n-m)}(f\circ g) \leq \overline{\tau}_h^{(b,q+n-m)}(f\circ g) < \infty$*,0¡* $\overline{\sigma}_l^{(y,n)}(g) \leq \sigma_l^{(y,n)}(g) < \infty$ *and* $\lambda_h^{(b,q+n-m)}(f\circ g) = \rho_l^{(y,n)}(g)$ *then*

$$\frac{\tau_h^{(b,q+n-m)}(f\circ g)}{\sigma_l^{(y,n)}(g)} \leq \varliminf_{r\to+\infty} \frac{\log^{[b-1]} \widehat{|h|}\left(|f\circ g|\left(\exp^{[q-m]} r\right)\right)}{\log^{[y-1]} \widehat{|l|}\left(|g|(r)\right)}$$
$$\leq \min\left\{\frac{\tau_h^{(b,q+n-m)}(f\circ g)}{\overline{\sigma}_l^{(y,n)}(g)}, \frac{\overline{\tau}_h^{(b,q+n-m)}(f\circ g)}{\sigma_l^{(y,n)}(g)}\right\}$$
$$\leq \max\left\{\frac{\tau_h^{(b,q+n-m)}(f\circ g)}{\overline{\sigma}_l^{(y,n)}(g)}, \frac{\overline{\tau}_h^{(b,q+n-m)}(f\circ g)}{\sigma_l^{(y,n)}(g)}\right\}$$
$$\leq \varlimsup_{r\to+\infty} \frac{\log^{[b-1]} \widehat{|h|}\left(|f\circ g|\left(\exp^{[q-m]} r\right)\right)}{\log^{[y-1]} \widehat{|l|}\left(|g|(r)\right)}$$
$$\leq \frac{\overline{\tau}_h^{(b,q+n-m)}(f\circ g)}{\overline{\sigma}_l^{(y,n)}(g)}.$$

Next we prove our theorems regarding (p, q)-th relative order and (p, q)-th relative type of f and its derivative f' with respect to g and its derivative g':

Theorem 3.21. *Let* f, $g \in \mathcal{A}(\mathbb{K})$. *Then for any positive integers* p *and* q

$$\rho_g^{(p,q)}(f) = \rho_g^{(p,q)}\left(f'\right) = \rho_{g'}^{(p,q)}(f) = \rho_{g'}^{(p,q)}\left(f'\right)$$

and

$$\lambda_g^{(p,q)}(f) = \lambda_g^{(p,q)}\left(f'\right) = \lambda_{g'}^{(p,q)}(f) = \lambda_{g'}^{(p,q)}\left(f'\right) .$$

Proof. Let $f(x) = \sum_{n=0}^{+\infty} a_n x^n \in \mathcal{A}(\mathbb{K})$ and $g(x) = \sum_{n=0}^{+\infty} b_n x^n \in \mathcal{A}(\mathbb{K})$. Since $|n| \leq 1$, so $(|n|)^n \leq |n|$. Therefore we have

$$|f'|(r) = \sup_{n\geq 1}\left\{|na_n| r^{n-1}\right\} = \frac{1}{r}\sup_{n\geq 1}\left\{|na_n| r^n\right\} \geq \frac{1}{r}\sup_{n\geq 0}\left\{|a_n|\left(|n|r\right)^n\right\},$$

$$\text{(3.37)} \qquad i.e., \ |f|\left(|n|r\right) \leq r|f'|(r).$$

Also for all $r > 0$ we know that

$$|f'|(r) \leq \frac{|f|(r)}{r} \ \{cf. \ [7, 18]\}. \tag{3.38}$$

Therefore from (3.37) and (3.38) ,we obtain for $r > 0$ that

$$\frac{f|(|n|r)}{r} \leq |f'|(r) \leq \frac{|f|(r)}{r}. \tag{3.39}$$

Now let $\alpha \in (0,1)$. Therefore replacing r by r^{α} and $b = \frac{1}{\alpha}$ in Lemma 2.2, we get that

$$|f|\left(r^{b\alpha}\right) \geq r^{a\alpha}|f|(r^{\alpha}) \geq r|f|(r^{\alpha}),$$

where a is a positive integer such that $a\alpha \geq 1$. Now for all sufficiently large values of r we obtain that

$$|f|(r) \geq r|f|(r^{\alpha}).$$

Therefore from (3.39) and above, we obtain for all large $r > 1$ that

$$|f|((|n|r)^{\alpha}) \leq \frac{f|(|n|r)}{r} \leq |f'|(r) \leq \frac{|f|(r)}{r} < |f|(r). \tag{3.40}$$

Therefore,

$$\overline{\lim_{r\to+\infty}} \frac{\log^{[p]} \widehat{|g|}(|f|((|n|r)^{\alpha}))}{\log^{[q]} r} \leq \overline{\lim_{r\to+\infty}} \frac{\log^{[p]} \widehat{|g|}(|f'|(r))}{\log^{[q]} r} \leq \overline{\lim_{r\to+\infty}} \frac{\log^{[p]} \widehat{|g|}(|f|(r))}{\log^{[q]} r},$$

$$i.e., \ \overline{\lim_{r\to+\infty}} \frac{\log^{[p]} \widehat{|g|}(|f|((|n|r)^{\alpha}))}{\log^{[q]} (|n|r)^{\alpha}} \cdot \frac{\log^{[q]} (|n|r)^{\alpha}}{\log^{[q]} r} \leq \rho_g^{(p,q)}\left(f'\right) \leq \rho_g^{(p,q)}(f). \tag{3.41}$$

Therefore when $q > 1$, we obtain from (3.41) that

$$\rho_g^{(p,q)}(f) \leq \rho_g^{(p,q)}\left(f'\right) \leq \rho_g^{(p,q)}(f) \ . \tag{3.42}$$

Further we $q = 1$, we get from from (3.41) that

$$\sigma\rho_g^{(p,q)}(f) \leq \rho_g^{(p,q)}\left(f'\right) \leq \rho_g^{(p,q)}(f) \ .$$

Now letting $\sigma \to 1-0$,we get from above that

$$\rho_g^{(p,q)}(f) \leq \rho_g^{(p,q)}\left(f'\right) \leq \rho_g^{(p,q)}(f) \ . \tag{3.43}$$

So from (3.42) and (3.43), it follow that

$$\rho_g^{(p,q)}\left(f'\right) = \rho_g^{(p,q)}(f). \tag{3.44}$$

Similarly if we replace g by g' , we obtain that

$$\rho_{g'}^{(p,q)}\left(f'\right) = \rho_{g'}^{(p,q)}(f) \ . \tag{3.45}$$

Further we obtain from (3.40) for all large $r > 1$ that

$$|g|\left(\left(|n|r\right)^{\alpha}\right) \leq |g'|\,(r) < |g|\,(r) \ .$$

Therefore from above we get that

$$\frac{\log^{[q]}\widehat{|f|}\left(|g|\left(\left(|n|r\right)^{\alpha}\right)\right)}{\log^{[p]}\left(|n|r\right)^{\alpha}}\cdot\frac{\log^{[p]}\left(|n|r\right)^{\alpha}}{\log^{[p]} r} \leq \frac{\log^{[q]}\widehat{|f|}\left(|g'|\,(r)\right)}{\log^{[p]} r} < \frac{\log^{[q]}\widehat{|f|}\left(|g|\,(r)\right)}{\log^{[p]} r} \ .$$

As $\lim\limits_{r\to+\infty}\frac{\log^{[p]}(|n|r)^{\alpha}}{\log^{[p]} r} \geq \alpha$, so we get from above that

$$\alpha \varliminf_{r\to+\infty}\frac{\log^{[q]}\widehat{|f|}\left(|g|\left(\left(|n|r\right)^{\alpha}\right)\right)}{\log^{[p]}\left(|n|r\right)^{\alpha}} \leq \varliminf_{r\to+\infty}\frac{\log^{[q]}\widehat{|f|}\left(|g'|\,(r)\right)}{\log^{[p]} r} \leq \varliminf_{r\to+\infty}\frac{\log^{[q]}\widehat{|f|}\left(|g|\,(r)\right)}{\log^{[p]} r},$$

$$i.e., \ \frac{\alpha}{\varlimsup\limits_{r\to+\infty}\frac{\log^{[p]}\widehat{|g|}(r)}{\log^{[q]}\widehat{|f|}(r)}} \leq \frac{1}{\varlimsup\limits_{r\to+\infty}\frac{\log^{[p]}\widehat{|g'|}(r)}{\log^{[q]}\widehat{|f|}(r)}} \leq \frac{1}{\varlimsup\limits_{r\to+\infty}\frac{\log^{[p]}\widehat{|g|}(r)}{\log^{[q]}\widehat{|f|}(r)}} \ .$$

So in view of Definition 1.6, it follows from above that

$$\frac{\alpha}{\rho_g^{(p,q)}(f)} \leq \frac{1}{\rho_{g'}^{(p,q)}(f)} \leq \frac{1}{\rho_g^{(p,q)}(f)} \ .$$

So letting $\sigma \to 1-0$,we have from above that

$$\rho_g^{(p,q)}(f) = \rho_{g'}^{(p,q)}(f) \ . \tag{3.46}$$

Thus the first part of the theorem follows from (3.44), (3.45) and (3.46).

In a similar manner one can easily establish the second part of the theorem. Hence the theorem follows. $\square$

The proof of the following theorem can be carried out as of and with the help of the Theorem 3.21, therefore we omit the details.

Remark 3.22. In the complex context, similar nature of results as proved in the first part of Theorem 3.21 were established by Lahiri et al [21].

Theorem 3.23. *Let f, $g \in \mathcal{A}(\mathbb{K})$. Then for any positive integers p and q, both greater than 1*

(1) $\sigma_g^{(p,q)}(f) = \sigma_g^{(p,q)}(f') = \sigma_{g'}^{(p,q)}(f) = \sigma_{g'}^{(p,q)}(f')$
(2) $\overline{\sigma}_g^{(p,q)}(f) = \overline{\sigma}_g^{(p,q)}(f') = \overline{\sigma}_{g'}^{(p,q)}(f) = v\overline{\sigma}_{g'}^{(p,q)}(f')$
(3) $\tau_g^{(p,q)}(f) = \tau_g^{(p,q)}(f') = \tau_{g'}^{(p,q)}(f) = \tau_{g'}^{(p,q)}(f')$ *and*
(4) $\overline{\tau}_g^{(p,q)}(f) = \overline{\tau}_g^{(p,q)}(f') = \overline{\tau}_{g'}^{(p,q)}(f) = \overline{\tau}_{g'}^{(p,q)}(f')$.

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